

Embeddings of Rank- 2 tori in Algebraic groups

Neha Hooda

Indian Statistical Institute, 7-S.J.S. Sansanwal Marg

New Delhi 110016, India

e-mail: neha.hooda2@ gmail.com

Abstract

Let k be a field of characteristic different from 2 and 3. In this paper we study connected simple algebraic groups of type A_2 , G_2 and F_4 defined over k , via their rank-2 k -tori. Simple, simply connected groups of type A_2 play a pivotal role in the study of exceptional groups and this aspect is brought out by the results in this paper. We refer to tori, which are maximal tori of A_n type groups, as unitary tori. We discuss conditions necessary for a rank-2 unitary k -torus to embed in simple k -groups of type A_2 , G_2 and F_4 in terms of the mod-2 Galois cohomological invariants attached with these groups. We calculate the number of rank-2 k -unitary tori generating these algebraic groups (in fact exhibit such tori explicitly). The results in this paper and our earlier paper ([8]) show that the mod-2 invariants of groups of type G_2, F_4 and A_2 are controlled by their k -subgroups of type A_1 and A_2 as well as the unitary k -tori embedded in them.

1 Introduction

The main aim of this paper is to investigate embeddings of rank-2 tori in groups of type G_2, F_4 and A_2 . In our earlier work ([8]), we studied k -embeddings of connected, simple algebraic groups of type A_1 and A_2 in simple groups of type G_2 and F_4 , defined over k , in terms of their respective mod-2 Galois cohomological invariants. We also showed that these groups are generated by their A_2 type k -subgroups. Owing to these results, importance of groups of type A_2 becomes evident in studying exceptional groups.

In the present paper, the mod-2 invariants of groups of type F_4, G_2 and A_2 are studied via the embeddings of certain rank-2 k -tori. To a simple, simply connected algebraic group G of type F_4, G_2 or A_2 defined over k , one attaches certain mod-2 Galois cohomological invariants, which are the Arason invariants of some Pfister forms attached to these groups. Let G be a group of type F_4 defined over k . Then there exists an Albert algebra A over k such that $G = \mathbf{Aut}(A)$, the full group of automorphisms of A . To any Albert algebra A , one attaches a certain reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$, for an octonion algebra C over k and $\Gamma = \text{Diag}(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$ ([21]). This defines two mod-2 invariants for $G = \mathbf{Aut}(A)$:

$$f_3(G) = f_3(A) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}),$$

$$f_5(G) = f_5(A) := e_5(n_C \otimes \langle\langle -\gamma_1^{-1}\gamma_2, -\gamma_2^{-1}\gamma_3 \rangle\rangle) \in H^5(k, \mathbb{Z}/2\mathbb{Z}),$$

where e_3 and e_5 are the Arason invariants of the respective Pfister forms and n_C is the norm form of C . We set $Oct(G) = Oct(A) := C$. Similarly, an algebraic group of type G_2 defined over k is precisely of the form $\mathbf{Aut}(C)$, for an octonion algebra C over k with norm form n_C , and this is classified by the Arason invariant $f_3(G) = e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$. Define $Oct(G) := C$. Finally, let G be a simple, simply connected group of type A_2 defined over k . To such a group G , one attaches an invariant $f_3(G) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$, the Arason invariant of a 3-fold Pfister form over k , which is the norm form of an octonion algebra C . We define $Oct(G) := C$ (see §2.5).

Let L, K be étale algebras over k of dimensions 3, 2 resp. and $T = \mathbf{SU}(L \otimes K, 1 \otimes \bar{})$, where $\bar{}$ denotes the non-trivial involution on K . Then T is a torus defined over k , referred to in the paper as the **K -unitary torus** associated with the pair (L, K) . For this torus, we let $q_T := \langle 1, -\alpha\delta \rangle = N_{k(\sqrt{\alpha\delta})/k}$, where $Disc(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$. Such tori are important as they occur as maximal tori in simple, simply connected groups of type A_2 and G_2 . We will be interested in conditions under which such tori embed in groups of type A_2, G_2 or F_4 defined over k . A unitary torus T will be called **distinguished** over k if q_T is hyperbolic over k , or equivalently, if $Disc(L) = K$. We shall see that the behaviour of the invariant f_3 for groups of type A_2 and G_2 is somewhat analogous to the behaviour of the invariant f_5 for groups of type F_4 , as far as embeddings of unitary tori in such groups is our concern.

Now we describe our main results. Let L, K be étale algebras over k of dimensions 3, 2 resp. and T be the **K -unitary torus** associated with the pair (L, K) . Let G be a simple, simply connected k -group of type A_2 or G_2 . We prove that if $T \hookrightarrow G$ over k , then q_T divides $f_3(G)$ (see Theorem 4.9). We show that G contains a distinguished maximal k -torus if and only if $f_3(G) = 0$ (see Theorems 3.2, 3.4). Similarly, for the groups of type F_4 we prove the following: Let G be a simple, simply connected k -group of type F_4 . If $T \hookrightarrow G$ over k , then q_T divides $f_5(G)$ (see Theorems 4.7). We show that a k -group G of type F_4 contains a distinguished k -torus if and only if $f_5(G) = 0$ (see Theorems 3.5). Let G be a k -group of type G_2 or a simply connected, simple group of type A_2 defined over k . We prove that a k -embedding $T \hookrightarrow G$ forces $K \subseteq Oct(G)$ (i.e., the Pfister form $\langle 1, -\alpha \rangle$ divides $f_3(G)$) (see Theorem 4.3). This fails to hold for groups of type F_4 , but holds in the special case when L has trivial discriminant (see Theorem 4.5). Let G be a k -group of type F_4 . Then the existence of a k -embedding $T \hookrightarrow G$ for a $K = k(\sqrt{\alpha})$ -unitary torus T implies that the Pfister form $\langle 1, -\alpha \rangle$ divides $f_5(G)$ (see Theorem 4.11).

Let L, K be étale algebras over k of dimensions 3, 2 resp. and let $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$, where $x \mapsto \bar{x}$ is the non-trivial k -automorphism of K . We define étale Tits process algebras J_1 and J_2 arising from the pair (L, K) to be L -isomorphic, if there exists a k -isomorphism $J_1 \rightarrow J_2$ which restricts to the subalgebra L of J_1 and J_2 . We establish a relation between $H^1(k, \mathbf{SU}(E, \tau))$ and the set of L -isomorphism classes of étale Tits process algebras arising from (L, K) (see Theorem 5.5). We also show that $H^1(k, \mathbf{SU}(E, \tau)) = 0$ if and only if all étale Tits process algebras arising from (L, K) are L -isomorphic to the étale Tits process $J(E, \tau, 1, 1)$ (see Theorem 5.6).

We study next the effect of the presence of a unitary torus T as above in groups of type A_2, G_2 and F_4 when $H^1(k, T) = 0$. We show that a k -group G of type G_2 contains a maximal k -torus $T \subset G$ such that $H^1(k, T) = 0$ if and only if the associated mod-2 invariant $f_3(G)$ vanishes, i.e., G splits (see Theorem 5.9). We derive a similar result for simple, simply connected k -groups of type A_2 : if such a group G has a maximal k -torus T with $H^1(k, T) = 0$,

then $f_3(G) = \text{Oct}(G)$ splits. The converse holds in the case when the group arises from a matrix algebra (see Theorem 5.7). For k -groups of type F_4 , we can prove a weaker result. Let G be a group of type F_4 (resp. G_2 or a simple, simply connected group of type A_2) defined over a field k . Assume that there is a k -embedding $T \hookrightarrow G$ for a unitary torus T associated to an étale pair (L, K) as above. If $H^1(\mathbf{U}(L \otimes K, 1 \otimes -)) = 0$ then $f_5(A) = 0$ (resp. $\text{Oct}(G)$ splits) (see Theorem 5.12).

In the final section of this paper, we compute the number of rank-2 k -tori (k a perfect field) generating simple, simply connected k -groups of type A_2 , G_2 , and F_4 , arising from division algebras and for k -subgroups of type D_4 of $\mathbf{Aut}(A)$, where A is an Albert division algebra. In fact, we explicitly exhibit such k -tori in each case. It seems likely that these numbers are minimal in each case. The numbers are mentioned in the table below.

| Table 1: Number of k -tori required for generation of groups | |
|--|--|
| Type of group | Number of rank-2 k -tori required for generation |
| A_2 | 2 |
| G_2 | 3 |
| D_4 | 3 |
| F_4 | 4 |

At the time of submission of this paper, we discovered the paper ([4]) by C. Bel, P. Gille and T.-Y. Lee, posted recently on the math arXiv. They have studied maximal tori in groups of type G_2 . Some of our results on groups of type G_2 partially match with results in this paper (see [4], Proposition 4.3.1., Corollary 4.4.2., Remarks 5.2.5. (b), Proposition 5.2.6 (i)), however the scope of our paper and methods of proofs are different.

Structure of the paper: In this paper, some of the results are valid over fields of general characteristics, however, for simplicity, we will work over a base field k of characteristic different from 2 and 3. We now proceed to describe the structure of the paper briefly:

§2 contains material, mainly preliminary in nature, on Albert algebras, octonion algebras, algebraic groups and the theory of unitary involutions on central simple algebras. We also fix some notation and terminology to be used in the rest of the paper. In §3, §4, we study the k -embeddings of rank-2 unitary k -tori in algebraic groups of type A_2 , G_2 and F_4 defined over k , in terms of the mod-2 Galois cohomological invariants attached to these groups. In §4.1, we discuss cohomology computations of unitary tori. §5 contains some applications of the cohomology computations in §4.1 to étale Tits process algebras. In §5.1 we mainly discuss consequences of presence of maximal k -tori T with $H^1(k, T) = 0$, in k -groups of type A_2 and G_2 , on the mod-2 invariants of such groups. In the final section §6, we prove the generation of k -groups of type A_2 , G_2 and F_4 (arising from division algebras) by their rank-2 unitary k -tori. We also compute the numbers of such tori, and exhibit such tori explicitly, in each of these cases.

2 Preliminaries

2.1 Notations

In this section we collect some notations which will be used in the paper. Let A be an associative algebra. By A^* we shall denote the group of units in A . In particular, for a field k , $k^* := k - \{0\}$. Let L be a finite étale extension of k . We will denote the norm and the trace maps by $N_{L/k}$, $T_{L/k}$ respectively. By $R_{L/k}^{(1)}(\mathbb{G}_m)$ we shall denote the k -torus $\text{Ker}\{N_{L/k} : L^* \rightarrow k^*\}$ of norm 1 elements in L and, when convenient, this torus will also be denoted by $\mathbf{L}^{(1)}$. Let G be an algebraic group defined over k . By $G(k)$ we denote the group of k -rational points in G . For a finite étale extension L of k the group of k -points of $\mathbf{L}^{(1)}$ will be denoted by $L^{(1)}$. We define an algebraic group G to be simple if G has no non-trivial proper connected normal subgroups. By $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle$ we shall mean the n -fold Pfister form $\langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$ over k . In the paper, unadorned tensor products will be understood to be over base fields and dimensions, when not specified, are over the base fields. For an object X defined over k , $X \otimes_k L$ will denote the base change $X \times_k L$ of X . By $K = k(\sqrt{\alpha})$ we denote the quadratic algebra $k[x]/(x^2 - \alpha)$ for $\alpha \in k^*$.

2.2 Unitary Involutions and their invariants

We need some results from the theory of unitary involutions on central simple algebras, for details we refer to ([6], [13]). Let k be a field with characteristic different from 2. Let K be a quadratic étale extension of k and let B be a central simple algebra of degree 3 over K with an involution σ of the second kind. Let T_B be the reduced trace map on B and Q_σ be the restriction of the trace form $Q : (x, y) \rightarrow T_B(xy)$ to $(B, \sigma)_+$, the k -space of symmetric elements of B . Then the decomposition of Q_σ is given by

Proposition 2.1. ([6], §4) *Let $K = k(\sqrt{\alpha})$. Then there exist $b, c \in k^*$ such that,*

$$Q_\sigma \cong \langle 1, 1, 1 \rangle \perp \langle 2 \rangle \cdot \langle\langle \alpha \rangle\rangle \cdot \langle -b, -c, bc \rangle.$$

In ([6], Theorem 15), it was proved that the 3-fold Pfister form $f_3(B, \sigma) := \langle\langle \alpha, b, c \rangle\rangle$ is an invariant for σ . The unitary involution σ is called **distinguished** if $f_3(B, \sigma)$ is hyperbolic (see [6], §4). For $B = M_3(K)$, up to automorphisms of (B, σ) , we have $\sigma = \text{Int}(a) \circ \tau$, where $\tau(x_{ij}) = (\overline{x_{ij}})^t$ with $a = \text{diag}(a_1, a_2, a_3) \in GL_3(k)$. By ([6], Prop. 2) we have,

$$Q_\sigma \cong \langle 1, 1, 1 \rangle \perp \langle 2 \rangle \cdot \langle\langle \alpha \rangle\rangle \cdot \langle a_1 a_2, a_1 a_3, a_2 a_3 \rangle.$$

In this case $f_3(B, \sigma) = \langle\langle \alpha, -a_1 a_2, -a_2 a_3 \rangle\rangle$. Hence, if σ is distinguished and K is a field, then $\langle a_1 a_2, a_1 a_3, a_2 a_3 \rangle_K \cong \langle 1, -1, -1 \rangle_K$.

The Arason invariant $e_n(\langle\langle a_1, a_2, \dots, a_n \rangle\rangle)$ of the n -fold Pfister form $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle$ is given by,

$$e_n(\langle\langle a_1, a_2, \dots, a_n \rangle\rangle) = (a_1) \cup (a_2) \cup \dots \cup (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z}),$$

where, for $a \in k^*$, (a) denotes the class of a in $H^1(k, \mathbb{Z}/2\mathbb{Z})$ (see [1], Pg. 453).

Proposition 2.2. ([6], Prop. 17) *Let B be as above. For every cubic étale k -subalgebra $L \subseteq B$, there is a distinguished involution σ such that $L \subseteq (B, \sigma)_+$.*

2.3 Albert Algebras

Octonion algebras and Pfister forms: Let C be an octonion algebra over k and let n_C denote its norm form. Then C is determined, up to isomorphism by n_C , which is a 3-fold Pfister form over k . Conversely, any 3-fold Pfister form is the norm form of a unique (up to isomorphism) octonion algebra over k . Recall that an octonion algebra C over k is **split** if and only if the associated norm form n_C is isotropic over k . Note that a Pfister form q over k is either hyperbolic over k or is k -anisotropic ([15], Chap. X, Theorem 1.7). Let C be an octonion algebra over k and $K \subseteq C$ be a quadratic étale subalgebra. Then K^\perp in C with respect to the norm form on C , has a rank-3 hermitian module structure over K . We record this below:

Proposition 2.3. ([9], §5) *Let C be an octonion algebra over k and $K \subseteq C$ be a quadratic étale subalgebra. Then $K^\perp \subseteq C$ has a rank-3 K -hermitian module structure given as follows: Let $K = k(\sqrt{\alpha})$, $\alpha \in k^*$. Define $h : K^\perp \times K^\perp \longrightarrow K$ by*

$$h(x, y) = N(x, y) + \alpha^{-1}N(\alpha x, y),$$

where $N(x, y)$ is the norm bilinear form of C and K acts on K^\perp from the left via the multiplication in C .

Let C be an octonion algebra over k with $x \mapsto \bar{x}$ as its canonical involution. Let $M_3(C)$ denote the algebra of 3×3 matrices with entries in C . Let $\Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$. Let

$$\mathcal{H}_3(C, \Gamma) = \{X \in M_3(C) \mid \Gamma^{-1} \bar{X}^t \Gamma = X\},$$

where, for $X = (x_{ij})$, $\bar{X} := (\bar{x}_{ij})$, and X^t is the transpose of X . The product $X \circ Y := \frac{1}{2}(XY + YX)$ defines a simple Jordan algebra structure on $\mathcal{H}_3(C, \Gamma)$, here $X, Y \in \mathcal{H}_3(C, \Gamma)$ and XY is the product of X and Y in $M_3(C)$. With this structure, $\mathcal{H}_3(C, \Gamma)$, is called a **reduced Albert algebra**.

Definition. A k -algebra A is called an *Albert algebra* if, over some field extension L of k , $A \otimes_k L$ is isomorphic to a reduced Albert algebra.

Reduced models for Albert algebras:

Let A be an Albert algebra over k . Then there exists, up to a k -isomorphism, a unique reduced Albert algebra $A_{\text{red}} := \mathcal{H}_3(C, \Gamma)$ over k , such that, for any extension L of k with $A \otimes_k L$ reduced, $A \otimes_k L \cong \mathcal{H}_3(C \otimes_k L, \Gamma)$ (see [20], §7). We call A_{red} the **reduced model** of A . By ([20], §4, §7), the isomorphism class of A determines A_{red} , and hence C , up to k -isomorphism. If $A_{\text{red}} = \mathcal{H}_3(C, \Gamma)$, we call C the **octonion algebra** of A , denoted by $\mathbf{Oct}(A)$.

Tits construction of Albert algebras: Let k be a field of characteristic different from 2 and 3. There are two rational constructions of Albert algebras due to Tits, which are exhaustive (though not exclusive). We briefly describe them below:

Tits's first construction

Let D be a central simple algebra of degree 3 over a field k and let N_D and T_D denote respectively the reduced norm and reduced trace maps on D . Let $\mu \in k^*$. Let $J(D, \mu) :=$

$D_0 \oplus D_1 \oplus D_2$, where $D_i = D$ for $i = 1, 2, 3$. Then there is a product on $J(D, \mu)$ with $(1, 0, 0)$ as identity and

$$N(x, y, z) = N_D(x) + \mu N_D(y) + \mu^{-1} N_D(z) - T_D(xyz), \quad x, y, z \in D,$$

as cubic norm, such that $J(D, \mu)$ is an Albert algebra over k , referred to as a **first Tits construction** Albert algebra. It is known that $J(D, \mu)$ is a division algebra if and only if $N(x, y, z)$ has no non-trivial zeros, if and only if μ is not a reduced norm from D ([13], Remark 39.15). By ([13], Prop. 40.5), an Albert algebra A over k is a first construction if and only if $\text{Oct}(A)$ is split over k .

Tits's second construction

Let K be a quadratic étale extension of k and B be a central simple algebra of degree 3 over K . Let σ be an involution of the second kind on B . Let $x \mapsto \bar{x}$ denote the non-trivial Galois automorphism of K/k . Let $(B, \sigma)_+$ be the k -subspace of B of σ -symmetric elements in B . Fix a unit u in $(B, \sigma)_+$ such that $N_B(u) = \mu \bar{\mu}$ for some $\mu \in K^*$. Let $J(B, \sigma, u, \mu) := (B, \sigma)_+ \oplus B$. There is a product on $J(B, \sigma, u, \mu)$ with $(1, 0)$ as identity and

$$N(a, x) = N_B(a) + \mu N_B(x) + \bar{\mu} N_B(\sigma(a)) - T_B(axu\sigma(a)), \quad a \in (B, \sigma)_+, \quad x \in B,$$

as cubic norm, such that $J(B, \sigma, u, \mu)$ is an Albert algebra over k , referred to as a **second Tits construction** Albert algebra. One knows that $J(B, \sigma, u, \mu)$ is a division algebra if and only if $N(a, x)$ has no non-trivial zeros, if and only if μ is not a reduced norm from B ([13], Theorem 39.18).

Theorem 2.1. ([11], Chapter. VI, Thm. 7) *Let k be a field of characteristic different from 2 and 3.. Let J, J' be finite dimensional Jordan algebras over k with identity such that J' is separable. If η is a norm isometry of J onto J' preserving identities, then η is an isomorphism of k -algebras.*

2.4 Algebraic groups

In this section, we collect some results on algebraic groups that we will need. We begin with some notation and terminology. For a finite dimensional k -algebra A , $\mathbf{Aut}(A)$ will denote the algebraic group $\text{Aut}_{\bar{k}}(A \otimes_k \bar{k})$ of algebra automorphisms defined over k . For a connected algebraic group G defined over k , the **k -rank** of G is defined as the dimension of a maximal k -split torus contained in G .

For a connected reductive group G over k , we say G is **k -isotropic** if there exists a non-central k -split torus in G and **k -anisotropic** otherwise. By the **type** of a connected reductive algebraic group G defined over k , we mean the Cartan-Killing type of its root system. By the **absolute rank** of an algebraic group G defined over k we mean the dimension of a maximal torus in G . By the **rank** of a torus T we mean its dimension.

Let A be a finite dimensional k -algebra and $S \subset A$ be a k -subalgebra. In the rest of the paper, we shall denote by $\mathbf{Aut}(A/S)$ the k -subgroup of $\mathbf{Aut}(A)$ consisting of all automorphisms of A which fix S pointwise and $\mathbf{Aut}(A, S)$ will denote the k -subgroup of $\mathbf{Aut}(A)$ mapping S to S .

Unitary groups and Unitary tori : Let K be a quadratic étale extension of a field k and let B be either a central simple K -algebra or an étale K -algebra in the sense of ([13], §18.A).

Assume that there is an involution σ on B of the second kind over K , i.e. σ restricts to K as the non-trivial k -automorphism of K . Let N_B denote the reduced norm map of the central simple algebra B or the norm map on the étale algebra B . We then define the algebraic groups $\mathbf{U}(B, \sigma)$ and $\mathbf{SU}(B, \sigma)$, by specifying the group of L -rational points, for any finite dimensional commutative k -algebra L , as follows :

$$\mathbf{U}(B, \sigma)(L) = \{x \in B \otimes L \mid x\sigma(x) = 1\}, \quad \mathbf{SU}(B, \sigma)(L) = \{x \in \mathbf{U}(B, \sigma) \mid N_B(x) = 1\}.$$

We note here that $\mathbf{SU}(B, \sigma)$ is a simple, simply connected algebraic groups of type A_2 defined over k when B is a central simple K -algebra of degree 3. When B is an étale algebra over K of rank n , then $\mathbf{U}(B, \sigma)$ and $\mathbf{SU}(B, \sigma)$ are tori defined over k (of rank resp. n and $n - 1$). We will denote $\mathbf{U}(B, \sigma)(k)$ by $U(B, \sigma)$ and $\mathbf{SU}(B, \sigma)(k)$ by $SU(B, \sigma)$.

Proposition 2.4. ([26], Prop. 6.3) *Let G be a connected reductive algebraic group defined over a perfect (infinite) field k , then G is k -anisotropic if and only if $G(k)$ contains no non-trivial unipotents .*

Proposition 2.5. ([28], Thm. 7.2.1, [19], Pg. 205) *Let A be an Albert algebra over a field k . Let $G = \mathbf{Aut}(A)$ be the algebraic group of automorphisms of A . Then G is a connected simple algebraic group of type F_4 defined over k . Moreover, G is k -isotropic if and only if A is reduced and $f_5(A) = 0$.*

Similarly, for groups of type G_2 , we have the following,

Proposition 2.6. ([28], §2.3, Thm. 2.3.5, [27], Prop. 17.4.2, 17.4.5) **(a)** *Let C be an octonion algebra over a field k and let $G = \mathbf{Aut}(C)$ be the algebraic group of automorphisms of C . Then G is a connected, simple algebraic group of type G_2 defined over k and G is either k -anisotropic or k -split. Moreover,*
(b) *G is k -anisotropic if and only if C is a division algebra, if and only if the norm form of C is k -anisotropic.*

The results below describe certain k -subgroups in k -groups of type G_2 and F_4 , these will be needed in the sequel.

Proposition 2.7. (Jacobson) *Let A be an Albert algebra defined over k and let $G = \mathbf{Aut}(A)$ denote the algebraic group of type F_4 associated with A . Let $L \subset A$ be a cubic étale subalgebra. Then the subgroup $\mathbf{Aut}(A/L)$ is a simply connected, simple group of type D_4 defined over k .*

Theorem 2.2. ([12], Thm. 9, [13], §39, Chap. IX) *Let A be an Albert algebra over k and let S be a 9-dimensional cubic separable Jordan subalgebra of A . The subgroup $\mathbf{Aut}(A/S)$ (resp. $\mathbf{Aut}(A, S)$) is a simply connected, simple algebraic group of type A_2 (resp. $A_2 \times A_2$) defined over k .*

Theorem 2.3. ([9], Thm. 3, Thm. 4, Thm. 5) *Let C be an octonion algebra over k and let S be a quadratic étale (resp. quaternion) subalgebra of C . Then the subgroup $\mathbf{Aut}(C/S)$ is a simply connected, simple group of type A_2 (resp. A_1) defined over k .*

Group actions : Let G be a group acting on a set X . Let $g \in G$ and H be a subgroup of G . We denote the set of fixed points of g in X by X^g and the set of fixed points of H in X by X^H . We will often use the following obvious fact

Lemma 2.1. *Let $g \in G$ and $h \in Z_G(g)$, the centralizer of g in G . Then h stabilizes X^g , i.e., maps X^g to itself. Consequently, for a subgroup H of G , its centralizer $Z_G(H)$ maps X^H to itself.*

Embeddings of A_2 in F_4 : Let B be a degree 3 central simple algebra over a quadratic étale extension K of k with an involution σ of the second kind. Let $A = J(B, \sigma, u, \mu)$ be a second Tits construction Albert algebra. Let $G = \mathbf{Aut}(A)$. Then G is a group of type F_4 over k . We have the following embedding of the special unitary group $\mathbf{SU}(B, \sigma)$ in G ,

$$\mathbf{SU}(B, \sigma) \hookrightarrow G \text{ via } p \mapsto \phi_p, \text{ where } \phi_p : (x, y) \mapsto (px\tau(p), py), \text{ for all } (x, y) \in A.$$

For a central simple algebra D over k , $\mathbf{SL}_1(D)$ denotes the algebraic group of norm 1 elements in D . When $A = J(D, \mu)$ is a first Tits construction Albert algebra, we have the embedding of $\mathbf{SL}_1(D)$ in $G = \mathbf{Aut}(A)$ given by,

$$\mathbf{SL}_1(D) \hookrightarrow G \text{ via } p \mapsto \phi_p, \text{ where } \phi_p : (x, y, z) \mapsto (x, yp, p^{-1}z), \text{ for all } (x, y, z) \in A.$$

2.5 Mod-2 invariants for groups of type A_2 , G_2 and F_4

Let G be a group of type G_2 defined over k . Then $G \cong \mathbf{Aut}(C)$ for a suitable octonion algebra C over k ([27], §17.4). Recall that C is determined by its **norm form** n_C , which is a 3-fold Pfister form over k . Hence the groups G over k of type G_2 are classified by the Arason invariant $f_3(G) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$, where $G \cong \mathbf{Aut}(C)$ as above.

Let G be a group of type F_4 defined over k . Then there exists an Albert algebra over k such that $G = \mathbf{Aut}(A)$, the full group of automorphisms of A ([27], §17.6). Let A be an Albert algebra over k and $A_{red} = \mathcal{H}_3(C, \Gamma)$ be the reduced model for A , as defined in §2.3. This defines two mod-2 invariants for $G = \mathbf{Aut}(A)$:

$$f_3(G) = f_3(A) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}),$$

$$f_5(G) = f_5(A) = e_5(n_C \otimes \langle 1, \gamma_1^{-1}\gamma_2 \rangle \otimes \langle 1, \gamma_2^{-1}\gamma_3 \rangle) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Proposition 2.8. *Let D be a degree 3 central division algebra or an Albert division algebra over k . Then D remains a division algebra over field extensions of degree coprime to 3.*

Proof. First, let D be a degree 3 central division algebra over k . By ([10], Exercise 9, Section 4.6), it follows that, for a field extension L of k of degree coprime to 3, $D \otimes L$ is a division algebra. When D is an Albert division algebra, the result follows from ([19], Cor., p. 205). \square

Octonion algebras for groups of type F_4 , G_2 and A_2

To a simple, simply connected algebraic group G of type F_4 , G_2 or A_2 defined over k , we associate an octonion algebra as follows. Let G be a k -group of type F_4 . Then $G \cong \mathbf{Aut}(A)$, for an Albert algebra A over k . Let $A_{red} = \mathcal{H}_3(C, \Gamma)$ be the reduced model for A , where C is

an octonion algebra over k and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$. Define $Oct(G) := Oct(A) = C$. Observe that $f_3(A) = 0$ if and only if $Oct(G)$ splits. Let G be a k -group of type G_2 . Then $G \cong \mathbf{Aut}(C)$, for an octonion algebra C over k . Define $Oct(G) := C$. Observe that $f_3(G) = 0$ if and only if G splits, if and only if $Oct(G)$ splits. Let G be a simple, simply connected k -group of type A_2 , then $G \cong \mathbf{SU}(B, \sigma)$, for some degree 3 central simple algebra B over a quadratic étale extension K of k , with an involution σ of the second kind. Define $Oct(G) := C$, where C is the octonion algebra determined by the 3-fold Pfister form $f_3(B, \sigma)$. The Arason invariant $f_3(B, \sigma)$ of the norm form of the octonion algebra of $(B, \sigma)_+$ is an invariant for G as well, we will denote this by $f_3(G)$, see ([8], Remark 2.8). Note that σ is distinguished if and only if $f_3(B, \sigma) = 0$ if and only if $Oct(G)$ splits. Moreover, these constructions are functorial and respect base change.

Lemma 2.2. (a) *Let G be a group of type F_4 defined over k . If G has k -rank ≥ 1 , then $f_5(G) = 0$. Moreover, if G has k -rank ≥ 2 , then G splits over k and $f_3(G) = 0 = f_5(G)$.*
 (b) *Let G be a simple, simply connected group of type G_2 or A_2 defined over k . If k -rank of $G \geq 1$, then $Oct(G)$ splits.*

Proof. (a) Let G be a group of type F_4 defined over k . If G has k -rank ≥ 1 then G is k -isotropic and by ([19], Pg. 205), $f_5(G) = 0$. If G has k -rank ≥ 2 then by ([31], Pg. 60), G splits over k . Hence A and thereby $Oct(A)$ splits over k ([27], Chap. 17, §17.6.4, [7], Pg. 164).

(b) Let G be a group of type G_2 defined over k . If k -rank of $G \geq 1$, then by ([27], Chap. 17, §17.4.2), G is k -split. Hence $Oct(G)$ is split ([27], Chap. 17, §17.4.5, [31], Pg. 60). Let G be a simple, simply connected group of type A_2 defined over k . If k -rank of $G \geq 1$ then G is k -isotropic and by ([8], Prop. 3.4), $Oct(G)$ splits. \square

Remark on notation : In the paper, to avoid surplus of notation, we will often confuse between the mod-2 invariants, f_3, f_5 (resp. f_3) of groups of type F_4 (resp. G_2), f_3 -invariant for simple, simply connected groups of type A_2 and the corresponding Pfister quadratic forms whenever no confusion is likely to arise. Let q be an n -fold Pfister form and $e_n(q) \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ be the Arason invariant of q . We will write $e_n(q) = 0$ to mean q is hyperbolic. Let q_1, q_2 be Pfister forms over k . We say q_2 divides q_1 over k if there exists a Pfister form q_3 over k such that $q_1 = q_2 \otimes q_3$ over k . If q_2 divides q_1 over k then by Theorem 4.6, q_2 is a subform of q_1 over k .

2.6 Étale algebras

Let k be a field of characteristic different from 2, 3 and L be an étale k -algebra of dimension n . Let $T : L \times L \rightarrow F$ be the bilinear form induced by the trace, $T(x, y) = T_{L/k}(xy)$, for $x, y \in L$. Let $\delta(L)$ denote the discriminant algebra of L over k .

Proposition 2.9. ([13], Prop. 18.24) *Let L be an étale k -algebra of dimension n . Then $\delta(L) = k[T]/(t^2 - d)$, where $d \in k^*$ represents the determinant of the bilinear form T .*

For the special case when L is a cubic étale k -algebra, by ([13], Prop. 18.25) we have,

Proposition 2.10. *Let L be an étale algebra of dimension 3 over k . There is a canonical k -isomorphism $L \otimes L \cong L \times L \otimes \delta(L)$.*

In this paper we will denote $\delta(L)$ by $Disc(L)$ and at times write also $Disc(L) = d$.

2.7 Maximal tori of special unitary groups

Let k be a field (of characteristic different from 2, 3) and K a quadratic field extension of k with the non-trivial k -automorphism $\bar{}$. Let V be a K -vector space of dimension n . Let h be a non-degenerate hermitian form on V . By ([29], Theorem 5.1) and Corollary 5.2, we have the following well known explicit description of maximal tori in a special unitary group of a non-degenerate hermitian space,

Theorem 2.4. (a) *Let k be a field and K a quadratic field extension of k . Let V be a K -vector space of dimension n with a non-degenerate hermitian form h . Let $T \subseteq \mathbf{U}(V, h)$ be a maximal k -torus. Then there exists an étale K -algebra E_T of dimension n over K , with an involution σ_h restricting to the non-trivial k -automorphism of K , such that $T = \mathbf{U}(E_T, \sigma_h)$.*
 (b) *Let $T \subset \mathbf{SU}(V, h)$ be a maximal k -torus. Then there exists an étale algebra E_T over K of dimension n , such that $T = \mathbf{SU}(E_T, \sigma_h)$*

Note that in Theorem 2.4, $E_T = Z_{\text{End}_K(V)}(T)$ with involution σ_h , where T is a maximal torus in $\mathbf{U}(V, h)$ or $\mathbf{SU}(V, h)$.

By ([29], Remark after Lemma 5.1) we have,

Lemma 2.3. *Let K be a quadratic étale extension of k . Let E be an étale algebra of dimension $2n$ over k containing K , equipped with an involution σ , restricting to the non-trivial k -automorphism of K . Let $L = E^\sigma = \{x \in E \mid \sigma(x) = x\}$. Then $E = L \otimes_k K$ and $(E, \sigma) = (L \otimes_k K, 1 \otimes \bar{})$, where $x \mapsto \bar{x}$ is the non-trivial k -automorphism of K .*

In view of the above lemma, $\dim(E^\sigma) = n$ over k . Let k be a field and L, K be étale k -algebras of k -dimension $n, 2$ resp. and $E = L \otimes K$. Then E is an étale algebra of dimension $2n$ over k . Let $\bar{}$ denote the non-trivial k -automorphism of K and τ the involution $1 \otimes \bar{}$ on E . We will refer to (E, τ) as the K -unitary algebra associated with the ordered pair (L, K) .

Lemma 2.4. *Let L, K be étale algebras of k -dimensions $n, 2$ resp. Let (E, τ) be the K -unitary algebra associated with (L, K) . Then $E^\tau = \{x \in E \mid \tau(x) = x\} = L$.*

Proof. Let K be a quadratic field extension. By Lemma 2.3, $\dim(E^\tau) = n = \dim(L)$. Since $L \subseteq E^\tau$, and the dimensions are equal, we have $L = E^\tau$. Let $K = k \times k$. Then $(E, \tau) = (L \times L, \epsilon)$, where ϵ is the switch involution on $L \times L$. Clearly $E^\tau = L$. \square

Let L, K be étale algebras of k -dimensions $n, 2$ resp. and (E, τ) be the K -unitary algebra associated with the pair (L, K) . We call the torus $\mathbf{SU}(E, \tau)$ as the K -unitary torus associated to the ordered pair (L, K) . With such a K -unitary torus T , we associate the quadratic form $q_T := \langle 1, -\alpha\bar{\delta} \rangle$, where $\text{Disc}(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$. Such tori are important as they occur as maximal tori in simple, simply connected groups of type A_{n-1} and G_2 .

We record below an evident, yet useful result:

Lemma 2.5. *Let K be a quadratic étale algebra over k and B be a degree 3 central simple algebra over K with an involution σ of the second kind. Let $L \subseteq (B, \sigma)_+$ be a cubic étale subalgebra. Let T be the K -unitary torus associated with the pair (L, K) . Then there exists a k -embedding $T \hookrightarrow \mathbf{SU}(B, \sigma)$.*

2.8 Maximal tori in groups of type G_2

The following result is well known (cf. for example, [30]), we supply a proof for convenience of the reader.

Proposition 2.11. *Let G be a group of type G_2 over k and T be a maximal k -torus of G . Then there exists étale algebras L, K of k -dimensions 3, 2 resp. such that T is the K -unitary torus associated to the pair (L, K) .*

Proof. Let G be as in hypothesis. Then there exists an octonion algebra C over k such that $G = \mathbf{Aut}(C)$. Let $T \subset G$ be a maximal k -torus in G .

Claim: There exists a quadratic étale subalgebra K of C such that $K = C^T$, the fixed points of the octonion algebra C under the action of T .

To see this, we may assume that the dimension $[C^t : k] = 4$ for all $t \in T(k)$. If not, then there exists $t \in T(k)$ such that $C^t = K$ is a quadratic étale subalgebra of C ([33], cf. also [14], Paragraph before Theorem 4 and [16]). Now T stabilizes, and hence, by a connectedness argument, fixes K pointwise. The claim then follows. Let $t \in T(k)$ be such that $C^t = Q$ for some quaternion subalgebra Q of C . Since T centralizes t , we see that $T \subseteq \mathbf{Aut}(C, Q)$. We write, by doubling process, $C = Q \oplus Qb$ for some $b \in Q^\perp$. Then by ([28], §2.1),

$$\mathbf{Aut}(C, Q) = \{\phi_{c,p}, c \in Q^*, p \in SL_1(Q) | \phi_{c,p}(x + yb) \rightarrow cxc^{-1} + (pcyc^{-1})b, \forall x, y \in Q\}.$$

By an easy computation it follows that for $c, c' \in Q^*$, $p, p' \in SL_1(Q)$ if $\phi_{c,p}\phi_{c',p'} = \phi_{c',p'}\phi_{c,p}$, then there exists $a \in k^*$ such that $cc' = ac'c$.

Claim: There exists $\phi_{c,p} \in T(k)$ such that $c \notin k^*$.

If not, then for all $\phi_{c,p} \in T(k)$ we have $c \in k^*$. Let $x \in Q$ be arbitrary and $y = 0$. Then, for any $\phi_{c,p} \in T$, $\phi_{c,p}(x) = x$. Thus $Q \subseteq C^T$ and hence $T \subseteq \mathbf{Aut}(C/Q)$, where $\mathbf{Aut}(C/Q)$ denotes subgroup of $\mathbf{Aut}(C)$ consisting of automorphisms of C which fixes Q pointwise. This is a contradiction, since T is a rank-2 torus and $\mathbf{Aut}(C/Q)$ is a simple group of type A_1 .

Thus there exists $\phi_{c,p} \in T(k)$ such that $c \notin k^*$. Since $\phi_{c,p} \in T$ is semisimple, c generates a quadratic étale subalgebra, $K := k(c)$ of Q . Let $\phi_{c',p'} \in T$. Since $\phi_{c',p'}$ commutes with $\phi_{c,p}$, we have $cc' = ac'c$ for some $a \in k^*$. Now, any element $\gamma \in K$ is a polynomial expression in c with coefficients from k , say,

$$\gamma = a_0 + a_1c + \dots + a_m c^m \text{ for } a_i \in k, m \in \mathbb{N}.$$

Now $\phi_{c',p'}(\gamma) = c'\gamma c'^{-1} = a_0 + aa_1c + \dots + aa^m a_m c^m \in K$. Hence $\phi_{c',p'}(\gamma) \in K$ for all $\gamma \in K$. Since $\phi_{c',p'}$ was chosen arbitrarily in T , we see that T stabilizes, and hence, fixes K pointwise. Hence $K \subseteq C^T$. Therefore $T \subseteq \mathbf{SU}(K^\perp, h)$, where h is the non-degenerate hermitian form on $K^\perp \subseteq C$ over K , induced by the norm bilinear form n_C (see [9], §5, cf. Prop. 2.3). Note that $\mathbf{SU}(K^\perp, h) = \mathbf{SU}(M_3(K), *_h)$, where $*_h$ is the involution on $M_3(K)$ given by $*_h(X) = h^{-1}\overline{X}^T h$. By Theorem 2.4, any maximal torus of $\mathbf{SU}(M_3(K), *_h)$ is of the form $\mathbf{SU}(E, *_h)$ for some six dimensional K -unitary algebra E over k . Hence $T = \mathbf{SU}(E, *_h)$ for some E as above. \square

In what follows, we will call a k -torus T as **distinguished** if there exists étale k -algebras L, K be of k -dimensions 3, 2 resp. such that $\text{disc}(L) = K$ and $T = \mathbf{SU}(E, \tau)$, where (E, τ) is the K -unitary algebra associated to the pair (L, K) . Also observe that when T is a distinguished k -torus, the associated quadratic form q_T splits over k . Note also that T has rank-2.

3 Distinguished tori in groups of type A_2 , G_2 and F_4

In this section we study embeddings of distinguished k -tori in simply connected, simple algebraic groups of type A_2 , G_2 and F_4 , defined over a field k , in terms of the mod-2 Galois cohomological invariants attached with these groups. We prove that a group G of type F_4 contains a distinguished k -torus if and only if $f_5(G) = 0$. A stronger version of this result holds for groups of type G_2 and A_2 . Let G be a simple, simply connected group of type G_2 or A_2 defined over k . We prove that $\text{Oct}(G)$ splits if and only if it G contains a distinguished (maximal) torus. We begin with,

Theorem 3.1. *Let T be a distinguished torus defined over k . Then T is isotropic over an odd degree extension of k .*

Proof. Let T be a distinguished torus over k . Then, by definition, there exists étale k -algebras L, K of k -dimensions 3, 2 resp. such that $\text{disc}(L) = K$ and $T = \mathbf{SU}(E, \tau)$, where (E, τ) is the K -unitary algebra associated to (L, K) . By Lemma 2.4, $L = E^\tau = \{x \in E \mid \tau(x) = x\}$. We divide the proof into three cases.

Case (i) $L = k \times k \times k$.

Since T is distinguished, we have $\text{disc}(L) = K = k \times k$. Hence $(E, \tau) \cong (L \times L, \varepsilon)$, where $\varepsilon : L \times L \rightarrow L \times L$ is given by $\varepsilon(x, y) = (y, x)$, the switch involution on $L \times L$. Now

$$SU(E, \tau) \cong \{(x, y) \in L \times L \mid (x, y)\varepsilon(x, y) = 1, (N_{L/k}(x), N_{L/k}(y)) = (1, 1)\} \cong L^{(1)} \cong k^* \times k^*,$$

where $L^{(1)}$ denotes the group of norm 1 elements of L . It follows that $\mathbf{SU}(E, \tau) \cong \mathbb{G}_m \times \mathbb{G}_m$ over k , and hence $T = \mathbf{SU}(E, \tau)$ splits over k in this case.

Case (ii) $L = k \times K$, K is a field.

Let $\bar{\varepsilon} : K \times K \rightarrow K \times K$ be given by $\bar{\varepsilon}(x, y) = (\bar{y}, \bar{x})$. Then $(E, \tau) = ((k \times K) \otimes K, 1 \otimes \bar{\varepsilon}) \cong (K \times (K \times K), (\bar{\cdot}, \bar{\varepsilon}))$. We have therefore,

$$\begin{aligned} SU(E, \tau) &\cong \{(x, y, z) \in K \times K \times K \mid (x, y, z)(\bar{x}, \bar{z}, \bar{y}) = (1, 1, 1), xyz = 1\} \\ &\cong \{(\bar{z}z^{-1}, \bar{z}^{-1}, z) \mid z \in K^*\} = \{(z^{-2}N(z), zN(z^{-1}), z) \mid z \in K^*\} \\ &\cong K^*. \end{aligned}$$

Hence $T = \mathbf{SU}(E, \tau) \cong R_{K/k}(\mathbb{G}_m)$ is isotropic over k .

Case (iii) L is a field.

Base changing to L we get, $L \otimes L \cong L \times K_0$ as L -algebras, where $K_0 = L \otimes \Delta$, and $\Delta = \text{Disc}(L)$ (see Prop. 2.10). By case (i) and (ii), it follows that $\mathbf{SU}(E, \tau) \otimes L$ is isotropic. Hence $T = \mathbf{SU}(E, \tau)$ is isotropic over L . \square

We now study the presence of distinguished k -tori in groups of type A_2 , G_2 and F_4 defined over k . We see that existence of such tori has a direct relation with the mod-2 invariants attached to these groups. We obtain as an immediate consequence of the above theorem the following,

Theorem 3.2. *Let G be a group of type G_2 over k . Then G splits over k (equivalently, $\text{Oct}(G)$ splits over k) if and only if there exists a maximal k -torus in G which is distinguished.*

Proof. Let $T \subseteq G$ be a distinguished maximal k -torus. By Theorem 3.1, T becomes isotropic over an odd degree extension, say M , of k . Hence M -rank of $G \geq 1$. Thus $\text{Oct}(G) \otimes M$ is split (Lemma 2.2). By Springer's theorem, $\text{Oct}(G)$ splits over k itself and consequently G is k -split. Conversely, suppose G splits over k . Let $L = k \times k \times k$ and $K = k \times k$ and $T = \text{SU}(L \otimes K, 1 \otimes \bar{})$. By case (i) of the proof of Theorem 3.1, $T \cong \mathbb{G}_m \times \mathbb{G}_m$ and $\mathbb{G}_m \times \mathbb{G}_m \hookrightarrow G$ over k as G is k -split. Hence T is the required distinguished k -torus. \square

A similar result holds for groups of type A_2 . We first recall,

Theorem 3.3. ([8], Prop. 3.4) *Let F be a quadratic étale k -algebra and B be a degree 3 central simple algebra over F with an involution σ of the second kind. Then σ is distinguished over k if and only if $\text{SU}(B, \sigma)$ becomes isotropic over an odd degree extension.*

Theorem 3.4. *Let G be a simple, simply connected group of type A_2 over k . Then $\text{Oct}(G)$ splits over k if and only if there exists a maximal k -torus in G which is distinguished.*

Proof. Let G be as in the hypothesis. Then $G \cong \text{SU}(B, \sigma)$ for some degree 3 central simple algebra B over a quadratic étale extension F of k with an involution σ of the second kind. Let $T \hookrightarrow G$ be a maximal k -torus which is distinguished. Then, by Theorem 3.1, T is isotropic over an odd degree extension M of k . Thus G is isotropic over M . Hence by Theorem 3.3, σ is distinguished over k . Hence, $f_3(B, \sigma) = 0$ and $\text{Oct}(G)$ is split over k . Conversely, if $\text{Oct}(G)$ is split over k , then $f_3(B, \sigma) = 0$ and hence σ is distinguished over k . By ([6], Theorem 16, pg. 317), $(B, \sigma)_+$ contains a cubic étale k - algebra L with F as its discriminant algebra. Let $E = L \otimes F$. Then $E \hookrightarrow B$ and σ restricted to E equals $\tau := 1 \otimes \bar{}$, where $\bar{}$ denotes the non-trivial k -automorphism of F . Hence $\text{SU}(E, \sigma)$ is a distinguished k -torus and, by Lemma 2.5, $\text{SU}(E, \tau) \hookrightarrow G \cong \text{SU}(B, \sigma)$ over k . \square

For groups of type F_4 we have the following,

Theorem 3.5. *Let A be an Albert algebra over k and $G = \text{Aut}(A)$. Then $f_5(A) = 0$ if and only if G contains a distinguished k -torus.*

Proof. Assume that G contains a distinguished k -torus T . Then by Theorem 3.1, T is isotropic over an odd degree extension M of k , hence G becomes isotropic over M . Therefore M -rank of $G \otimes M \geq 1$ and $f_5(A \otimes M) = f_5(A) \otimes M = 0$ (Lemma 2.2). By Springer's theorem $f_5(A) = 0$. Conversely, if $f_5(A) = 0$, by ([13], Prop. 40.7), $A \cong J(B, \sigma, u, \mu)$ for a central simple algebra B over a quadratic étale extension F of k , with a distinguished involution σ . Since σ is distinguished, by Theorem 3.4 there exists a k -embedding of a distinguished k -torus T in $\text{SU}(B, \sigma)$. Now $\text{SU}(B, \sigma) \hookrightarrow G$ over k (see §2.4). Hence $T \hookrightarrow G$ over k and T is distinguished. \square

As a consequence of the above theorem, we have an alternative proof of ([8], Theorem 3.4).

Corollary 3.1. *Let A be an Albert algebra over k and $G = \text{Aut}(A)$. Then $f_5(A) = 0$ if and only if there exists a k -embedding $\text{SU}(B, \sigma) \hookrightarrow G$ for some degree 3 central simple algebra B with center a quadratic étale k -algebra F and with a distinguished involution σ .*

Proof. Suppose $\text{SU}(B, \sigma) \hookrightarrow G$ over k for (B, σ) as in the hypothesis. Since σ is distinguished, by Theorem 3.4, there exists a k -embedding $T \hookrightarrow \text{SU}(B, \sigma)$ for a distinguished k -torus T . Hence $T \hookrightarrow \text{SU}(B, \sigma) \hookrightarrow G$ over k . Therefore, by Theorem 3.5, $f_5(A) = 0$. The proof of the converse follows exactly along the same lines as in the proof of Theorem 3.5. \square

4 Embeddings of rank-2 tori in A_2 , G_2 and F_4

It turns out that embeddings of unitary tori in groups of type A_2 , G_2 and F_4 are intricately linked to the mod-2 invariants of these groups. We discuss this below. First we fix some terminology which will be used in the sequel.

Groups arising from division algebras: Let G be a simple, simply connected k -group of type A_2 . We will refer to G as arising from a division algebra if either $G \cong \mathbf{SU}(D, \sigma)$ for some degree 3 central division algebra D over a quadratic field extension F of k , with an involution σ of the second kind or $G \cong \mathbf{SL}_1(D)$ for some degree 3 central division algebra D over k . Let G be a k -group of type F_4 . We will refer to G as arising from a division algebra if $G \cong \mathbf{Aut}(A)$, where A is an Albert division algebra over k . Let G be a k -group of type G_2 . We will refer to G as arising from a division algebra if $G \cong \mathbf{Aut}(C)$, where C is an octonion division algebra over k .

Theorem 4.1. *Let G be a simple, simply connected group of type A_2 or F_4 defined over k , arising from a division algebra over k . Then,*

- (1) $G(k)$ contains no non-trivial involution over k .
- (2) There does not exist any rank-1 torus T over k such that $T \hookrightarrow G$ over k .
- (3) G is k -anisotropic.

Moreover, these conditions hold over any field extension of k of degree coprime to 3.

Proof. First we prove (1). Recall that an involution in a group is an element of order at most 2. Let G be a simple, simply connected group of type A_2 , arising from a division algebra D over k . Let $Z(D)$ denote the center of D . Then $[D : Z(D)] = 9$. Let $\theta \in G(k) \subseteq D^*$ be an involution. Then $\theta^2 = 1$ and $N_D(\theta) = 1$. Since $\theta^2 = 1$, θ generates the field extension $k(\theta)$ of k of degree ≤ 2 over $Z(D)$. Since the dimension $[D : Z(D)] = 9$, it follows that $\theta \in Z(D)$. Since $\theta^2 = 1$ and $\theta \in Z(D)$, $\theta = 1$ or -1 ($Z(D)$ is a field). Since $N_D(\theta) = 1$ we have $\theta = 1$. Hence $G(k)$ does not contain any non-trivial involutions. When G is a group of type F_4 , the implication follows from a theorem of Jacobson ([11], Chap. IX, Theorem 9). Moreover, let M be any field extension of k of degree coprime to 3. As seen above, if $G(M)$ contains a non-trivial involution, then $G \otimes M$ cannot arise from a division algebra. By Proposition 2.8, G cannot arise from a division algebra.

We now prove (2). Suppose there exists a rank-1 torus T over k such that $T \hookrightarrow G$ over k . Necessarily, $T = K^{(1)}$, the norm torus of a quadratic extension K/k ([32], Chap.II, §IV, Example 6). But then T splits over K , which in turn implies that G becomes isotropic over K . Suppose G is a group of type F_4 over k . Then $G = \mathbf{Aut}(A)$ for some Albert algebra A over k . Since G becomes isotropic over K , $A \otimes K$ is reduced (see Prop. 2.5). Hence G does not arise from a division algebra over k , since no extension of degree coprime to 3 can reduce an Albert division algebra (Proposition 2.8). This is a contradiction. Now suppose G is a group of type A_2 over k . Since G becomes isotropic over K , by ([31], Table of Tits indices), $G \otimes K$ does not arise from a division algebra over K . By Proposition 2.8, G does not arise from a division algebra over k , a contradiction. Moreover, let M be any field extension of k of degree coprime to 3. Suppose there exists a rank-1 torus T over M such that $T \hookrightarrow G$ over M . Then, as seen above, G does not arise from a division algebra over M . Hence by Proposition 2.8, G does not arise from a division algebra over k . This is a contradiction. The proof of (3) follows from ([31], Remark on Page 61, Table of Tits indices). \square

Theorem 4.2. *Let A be an Albert algebra over k and $G = \mathbf{Aut}(A)$. Then the following are equivalent.*

- (a) $f_3(A) = 0$ (i.e, $\text{Oct}(G)$ is split).
- (b) *There exists a cubic étale k -algebra L of trivial discriminant such that $\mathbf{L}^{(1)} \hookrightarrow G$ over k .*
- (c) $\mathbf{SL}_1(D) \hookrightarrow G$ over k , for a degree 3 central simple algebra D over k .

Proof. Let $f_3(A) = 0$. Then, by ([13], Prop. 40.5), A is a first Tits construction and $A \cong J(D, \mu)$, where D is a degree 3 central simple algebra over k . If D is split, let $L = k \times k \times k$ and if D is a division algebra, let L a cubic cyclic extension of k such that $L \subseteq D_+$ (This is possible by Wedderburn's Theorem [13], Pg. 303, 19.2). In either case, since $\mathbf{SL}_1(D) \hookrightarrow G$ (see §2.4), $\mathbf{L}^{(1)} \hookrightarrow G$ over k . Hence (a) \Rightarrow (b) and (a) \Rightarrow (c) follows.

For the proof of (b) \Rightarrow (a), let $\mathbf{L}^{(1)} \hookrightarrow G$ over k , where L is a cubic étale k -algebra of trivial discriminant. Clearly $L \cong k \times k \times k$ or L is a cubic cyclic field extension of k . If $L \cong k \times k \times k$ then $\mathbf{L}^{(1)} \cong \mathbb{G}_m \times \mathbb{G}_m$. Hence the k -rank of $G \geq 2$ and, by Lemma 2.2, $f_3(A) = 0$. Let L be a cubic cyclic field extension of k . Observe that $\mathbf{L}^{(1)} \otimes L \cong \mathbf{E}^{(1)}$, where $E = L \otimes L$. By Proposition 2.10, $L \otimes L \cong L \times L \times L$ and hence $\mathbf{E}^{(1)}$ is an L -split torus of rank-2, embedding in $G \otimes L$. Hence the L -rank of $G \otimes L \geq 2$ and thus, by Lemma 2.2, $f_3(A \otimes L) = 0$. By Springer's theorem, $f_3(A) = 0$. We now prove (c) \Rightarrow (a). Let $\mathbf{SL}_1(D) \hookrightarrow G$ over k , where D is a degree 3 central simple algebra over k . If D is a division algebra, choose a cubic separable extension L over k , $L \subseteq D$. Now,

$$D \otimes_k L \cong M_3(L) \text{ and } \mathbf{SL}_1(D \otimes_k L) \cong \mathbf{SL}_3 \hookrightarrow G \otimes L.$$

Hence $G \otimes L$ has L -rank ≥ 2 . By Lemma 2.2, $\text{Oct}(A)$ splits over L . Since $[L : K] = 3$, by Springer's theorem, $\text{Oct}(A)$ must split over k and $f_3(A) = 0$. In the case when D is split, $\mathbf{SL}_1(D) \cong \mathbf{SL}_3 \hookrightarrow G$ over k . Hence G is split over k and $f_3(A) = 0$. \square

Lemma 4.1. *Let $L = k \times K_0$ be a cubic étale algebra over k , where K_0 is a quadratic étale extension of k . Let $K = k \times k$ and T be the K -unitary torus associated with the pair (L, K) . Then $T \cong R_{K_0/k}(\mathbb{G}_m)$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$. Note that

$$(L \otimes K, 1 \otimes \bar{}) \cong (L \times L, \varepsilon) \cong ((k \times K_0) \times (k \times K_0), \varepsilon),$$

where $\varepsilon : L \times L \mapsto L \times L$ is the switch involution. Hence

$$\mathbf{SU}(E, \tau) \cong \mathbf{SU}((k \times K_0) \times (k \times K_0), \varepsilon).$$

For $((a, x), (b, y)) \in (k \times K_0) \times (k \times K_0)$ we have,

$$((a, x), (b, y))\varepsilon((a, x), (b, y)) = (((a, x), (b, y))((b, y), (a, x))) = ((ab, xy), (ba, yx)), \text{ and}$$

$$N_{E/K}((a, x), (b, y)) = N_{(k \times K_0) \times (k \times K_0)/(k \times k)} = (a.N_{K_0/k}(x), b.N_{K_0/k}(y)).$$

Hence,

$$\mathbf{SU}(E, \tau) \cong \{((a, x), (a^{-1}, x^{-1})) \in (k \times K_0) \times (k \times K_0) \mid a.N_{K_0/k}(x) = 1\} \cong K_0^*.$$

From this it follows that $\mathbf{SU}(E, \tau) \cong R_{K_0/k}(\mathbb{G}_m)$. \square

Lemma 4.2. *Let $L = k \times k \times k$ and K be a quadratic étale extension of k . Let T be the K -unitary torus associated with the pair (L, K) . Then $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$. It is immediate that $(E, \tau) \cong (K \times K \times K, (\bar{}, \bar{}, \bar{}))$. Hence,

$$SU(E, \tau) \cong \{(x, y, z) \in K \times K \times K \mid x\bar{x} = y\bar{y} = z\bar{z} = 1, xyz = 1\} \cong K^{(1)} \times K^{(1)}.$$

It follows that $\mathbf{SU}(E, \tau) \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. □

Theorem 4.3. (a) *Let G be a k -group of type G_2 or a simply connected, simple group of type A_2 . Let L, K be étale algebras of dimension 3, 2 resp. and T be the K -unitary torus associated with the pair (L, K) . Suppose there exists a k -embedding $T \hookrightarrow G$. Then $K \subseteq \text{Oct}(G)$.*

(b) *If G is a k -group of type F_4 or a simply connected, simple group of type A_2 arising from a division algebra and $T \hookrightarrow G$ over k , then L must be a field extension.*

Proof. Let (E, τ) and T be the K -unitary algebra and torus resp. associated with the pair (L, K) . By definition $T = \mathbf{SU}(E, \tau)$. For the assertion **(a)**, we divide the proof into two cases.

Case 1: $L = k \times K_0$ for some quadratic étale extension K_0 of k .

Let $K = k \times k$. By Lemma 4.1, $T \cong R_{K_0/k}(\mathbb{G}_m) \hookrightarrow G$. Therefore, the k -rank of $G \geq 1$. Thus by Lemma 2.2, $\text{Oct}(G)$ is split. When K is a field extension, base changing to K and applying the same argument, it follows that $\text{Oct}(G) \otimes K$ is split. Hence $K \subseteq \text{Oct}(G)$ ([5], Lemma 5).

Case 2: L is a field extension.

Base changing to L , by Proposition 2.10, we have, $L \otimes L \cong L \times K_0$ for $K_0 = L \otimes \Delta$, where Δ is the discriminant algebra of L over k . By case 1, $K \otimes L \subseteq \text{Oct}(G) \otimes L$. Therefore if $K = k \times k$, $\text{Oct}(G) \otimes L$ is split and by Springer's theorem, $\text{Oct}(G)$ splits and $K \subseteq \text{Oct}(G)$. Hence we may assume that K is a field. Then $K \otimes L$ is a cubic field extension of K and

$$(\text{Oct}(G) \otimes L) \otimes_L (L \otimes K) \cong \text{Oct}(G) \otimes L \otimes K \cong (\text{Oct}(G) \otimes K) \otimes_K (K \otimes L)$$

is split, since $K \otimes L \subseteq \text{Oct}(G) \otimes L$. Hence $(\text{Oct}(G) \otimes K)$ is split over the cubic extension $(K \otimes L)$ of K . Therefore by Springer's theorem, $\text{Oct}(G) \otimes K$ is split. Hence $K \subseteq \text{Oct}(G)$ ([5], Lemma 5).

Now we prove **(b)**. Let G be a k -group of type F_4 or A_2 as in the hypothesis and let $T \hookrightarrow G$ over k , where T is the K -unitary torus associated to the pair (L, K) as in the hypothesis. Assume that L is not a field. Let $L = k \times K_0$ for some quadratic field extension K_0 of k . If $K = k \times k$ then, as in the proof of case 1, G is k -isotropic. Therefore, by Theorem 4.1, G cannot arise from a division algebra. Let K be a field extension. By an easy calculation we see that $T \otimes K_0 = \mathbf{SU}(E \otimes K_0, \tau) \cong \mathbf{M}^{(1)} \times \mathbf{M}^{(1)}$, where $M = (K \otimes K_0)$. Note that $M^{(1)} \times M^{(1)}$ contains the involution $(-1, 1)$ defined over K_0 . Hence $G(K_0)$ contains a non-trivial involution. Therefore, by Theorem 4.1, G cannot arise from a division algebra. In the case when $L = k \times k \times k$, by Lemma 4.2, $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. Again $K^{(1)} \times K^{(1)}$ contains the involution $(-1, 1)$ defined over k . Hence $G(k)$ contains a non-trivial involution. Therefore, by Theorem 4.1, G cannot arise from a division algebra. Hence **(b)** follows. □

Remark 4.4. (1) For k -groups of type G_2 , **(b)** fails to hold. To see this, let $L = k \times k \times k$ and K be a quadratic field extension of k . Let T be the K -unitary torus associated with the

pair (L, K) . By Lemma 4.2, $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. Such a torus embeds in a k -group of type G_2 arising from a division algebra (see [28], §2.1).

(2) For k -groups of type F_4 , (a) fails to hold. Let C be an octonion division algebra over k . Let $\Gamma = \text{diag}(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Then $C = \text{Oct}(G)$ (see §2.3). Let $F \subseteq C$ be a quadratic subfield. By ([24], §1, Thm. 1.1), there exists an isomorphism of Jordan algebras $\mathcal{H}_3(C, \Gamma) \cong J(M_3(F), *_\Gamma, V, \mu)$, where $*_\Gamma(X) = \Gamma^{-1} \bar{X}^t \Gamma$, $V \in GL_3(F)$ with $*_\Gamma(V) = V$ and $\det V = \mu \bar{\mu}$ for some $\mu \in F^*$. Let $L = k \times F$. Note that $L \subseteq M_3(F)$ as a k -subalgebra (via the embedding $(\gamma, x) \rightarrow \text{diag}(\gamma, x, x)$, $\gamma \in k, x \in F$). Since $*_\Gamma$ is a distinguished involution on $M_3(F)$ ([6], Theorem 16), by ([6], Cor. 18), it follows that $L \hookrightarrow (M_3(F), *_\Gamma)_+$ over k . Let T be the F -unitary torus associated with the pair (L, F) . Then $T \hookrightarrow \mathbf{SU}(M_3(F), *_\Gamma) \hookrightarrow G$ over k (see §2.4). By case (ii) of Theorem 3.1, $T \cong R_{F/k}(\mathbb{G}_m)$. Hence $R_{F/k}(\mathbb{G}_m) \hookrightarrow G$ over k . Now consider $K = k \times k$. By Lemma 4.1, $\mathbf{SU}(L \otimes K, \tau) \cong R_{F/k}(\mathbb{G}_m)$. Hence $\mathbf{SU}(L \otimes K, \tau) \hookrightarrow G$ over k but K does not embed in $C = \text{Oct}(G)$, since C is a division algebra.

However we have the following,

Theorem 4.5. *Let G be a group of type F_4 defined over k . Let K be a quadratic étale k -algebra and L be a cubic étale k -algebra with trivial discriminant. Let T be the K -unitary torus associated with the pair (L, K) . Suppose $T \hookrightarrow G$ over k . Then $K \subseteq \text{Oct}(G)$.*

Proof. Let L be as in the hypothesis. When $K = k \times k$, we have $(L \otimes K, \tau) \cong (L \times L, \epsilon)$, where $\epsilon(x, y) = (y, x)$ for all $(x, y) \in L \times L$. Hence $T \cong \mathbf{L}^{(1)}$. By Theorem 4.2, $\text{Oct}(G)$ splits and hence $K \subseteq \text{Oct}(G)$. When K is a field extension, base changing to K we see that $\text{Oct}(G) \otimes K$ splits. Hence $K \subseteq \text{Oct}(G)$ ([5], Lemma5). \square

We now prove a factorization result for the mod-2 invariant $f_5(G)$ associated to an algebraic group G of type F_4 defined over k , given an embedding of a rank-2 K -unitary torus in G . Let L, K be étale algebras of dimension 3, 2 resp. and let T be the K -unitary torus associated with the pair (L, K) . Recall that with T , we associate the quadratic form $q_T := \langle 1, -\alpha\delta \rangle$, where $\text{Disc}(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$.

We need a classical factorization result for quadratic forms. Let q be a quadratic form over a field F . Let $D_F(q)$ denote the set of values in F^* represented by q and $[D_F(q)]$ be the subgroups of F^* generated by $D_F(q)$. When q is a Pfister form $[D_F(q)] = D_F(q)$ ([15], Chap. X, Theorem 1.8).

The following result on quadratic forms is essential for what follows.

Theorem 4.6. ([15], Chap. IX, Pg. 305, Chap. X, Cor. 4.13) *For any quadratic form ϕ and any anisotropic quadratic form γ over k , the following are equivalent,*

- (i) $\phi \subseteq \gamma$ (i.e. ϕ is isometric to a subform of the form γ over k).
- (ii) $D_K(\phi) \subseteq D_K(\gamma)$ for any field $K \supseteq k$, where $D_K(\phi)$ denotes the set of values in K^* represented by ϕ . Moreover, if ϕ and γ are both Pfister forms, then the above conditions are also equivalent to
- (iii) $\gamma = \phi \otimes \tau$ for some Pfister form τ over k (In this case we will call ϕ as a factor of γ).

We can now prove:

Theorem 4.7. *Let A be an Albert algebra over k and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale k -algebra and L be a cubic étale k -algebra with discriminant δ . Let T be the K -unitary torus associated with the pair (L, K) . Suppose $T \hookrightarrow G$ over k . Then $f_5(A) = q_T \otimes \gamma$ for some 4-fold Pfister form γ over k .*

Proof. Let $G = \mathbf{Aut}(A)$ be as in the hypothesis and let $T \hookrightarrow G$ over k .

Claim: $D_M(q_T \otimes M) \subseteq D_M(f_5(A) \otimes M)$ for all field extensions M of k .

Let $F = k(\sqrt{\alpha\delta})$. Then $N_{F/k} = q_T$. Note that, $N_{F/k} \otimes M = N_{F \otimes M/M}$ and $f_5(A) \otimes M = f_5(A \otimes M)$. If $N_{F \otimes M/M}$ is hyperbolic over M , then $\alpha = \delta M^{*2}$ and hence $T \otimes M$ is a distinguished torus. Therefore, by Theorem 3.5, $f_5(A \otimes M) = 0$ and the claim follows trivially. We may therefore assume both $N_{F \otimes M/M}$ and $f_5(A \otimes M)$ are anisotropic. Hence $K' := F \otimes M$ is a field extension of M . Now further base changing to $K' \cong M \otimes_M K'$ we get,

$$(T \otimes M) \otimes_M K' \cong T \otimes_M K' \hookrightarrow \mathbf{Aut}((A \otimes_k M) \otimes_M K'). \quad (*)$$

Since $\alpha = \delta K'^{*2}$, $T \otimes_M K'$ is a distinguished torus. Taking K' as the base field and applying Theorem 3.5 to the embedding $(*)$ we get, $f_5((A \otimes_k M) \otimes_M K') = 0$. Now, since K' over M is a finite field extension and $f_5(A \otimes_k M) \otimes_M K'$ is split, we have, by Theorem ([15], Chap. VII, Cor. 4.4), $N_{K'/M}(K'^*) \subseteq D_M(f_5(A \otimes_k M))$. Since $N_{K'/M}(K'^*) = D_M(q_T \otimes M)$, we have $D_M(q_T \otimes M) \subseteq D_M(f_5(A) \otimes M)$ for all extensions M of k . Hence by Theorem 4.6, $N_{F/k}$ is isometric to a subform of $f_5(A)$ and we have, $f_5(A) = q_T \otimes \gamma$, for some 4-fold Pfister form γ over k . \square

Remark 4.8. 1) Note that the converse of the above theorem fails to hold.

Let C denote the octonion division algebra represented by the 3-fold (anisotropic) Pfister form $\langle 1, -x \rangle \otimes \langle 1, -y \rangle \otimes \langle 1, -z \rangle$ over $k = \mathbb{C}(x, y, z, w)$. Let $F \subset C$ be a quadratic subfield and let $h = \text{diag}(h_1, h_2, h_3)$ denote the hermitian form on $F^\perp \subset C$ induced by the norm bilinear form (see [9], §5, cf. Prop. 2.3). Consider the Albert algebra $A := J(M_3(F), *_h, 1, \mu)$ where $*_h(X) = h^{-1} \overline{X}^t h$ and $\mu \in F$ satisfies $\mu \overline{\mu} = 1$. Let $G = \mathbf{Aut}(A)$. Then $\text{Oct}(G) = C$ ([24], §1, Theorem 1.1). By ([8], Lemma 3.2), $f_5(A) = n_C \otimes \langle -1, -1 \rangle$. Since -1 is a square in k , we have $f_5(A) = 0$. Let $K = k(\sqrt{w})$ and let L be any cubic cyclic field extension of k . Let T be the K -unitary torus associated with the pair (L, K) . Since $-w$ is not represented by n_C , $K \not\subset C$. Hence, by Theorem 4.5, T cannot embed in G over k , however q_T divides $f_5(A)$.
2) Let q_T be as in the hypothesis of Theorem 4.7. Note that q_T does not divide $f_3(G)$ in general. We use the construction as in the case (2) of Remark 4.4. Let C be an octonion division algebra. Let $\Gamma = \text{diag}(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Note that $\text{Oct}(G) = C$. Let $F \subseteq C$ be a quadratic subfield and $L = k \times F$. Let T be the F -unitary algebra associated with the pair (L, F) . Then, as in the case (2) of Remark 4.4, $T \hookrightarrow G$ over k . Since $\text{Disc}(L) = F$, we have $\alpha = \delta \bmod k^{*2}$. Hence the Pfister form $q_T = \langle 1, -\alpha \delta \rangle \cong \langle 1, -1 \rangle$ and q_T does not divide $f_3(G)$, since C is a division algebra.

On exactly similar lines we can derive a necessary condition for a rank-2 unitary torus to embed in a connected simple algebraic group of type A_2 or G_2 :

Theorem 4.9. *Let G be a simple, simply connected k -group of type A_2 or G_2 . Let $C := \text{Oct}(G)$ and n_C denote the norm form of C . Let $K = k(\sqrt{\alpha})$ be a quadratic étale k -algebra*

and L be a cubic étale k -algebra with discriminant δ . Let T be the K -unitary torus associated with the pair (L, K) . Suppose there exists a k -embedding $T \hookrightarrow G$. Then $n_C = q_T \otimes \gamma$ for some two fold Pfister form γ over k .

Proof. By Theorems 3.2, 3.4, one sees that if T is distinguished then C splits. Now using same arguments as in the proof of Theorem 4.7, we get the desired result. \square

Remark 4.10. Note that the converse of the above theorem fails to hold.

1) Let $*$ denote the unitary involution $*(X) = \overline{X}^t$ on $M_3(\mathbb{C})$ and let $G = \mathbf{SU}(M_3(\mathbb{C}), *)$. Let $C = \text{Oct}(G)$. Then $n_C = \langle 1, 1 \rangle \otimes \langle 1, 1 \rangle \otimes \langle 1, 1 \rangle$ (see §2.2). Hence C is the unique octonion an division algebra over \mathbb{R} . Take $K = \mathbb{R} \times \mathbb{R}$ and $L = \mathbb{R} \times \mathbb{C}$. Let T be the K -unitary torus associated with the pair (L, K) . Since C is a division algebra, $K = \mathbb{R} \times \mathbb{R} \not\subset C$. Hence, by Theorem 4.3, T does not embed in G over \mathbb{R} but the quadratic form $q_T = \langle 1, 1 \rangle$ associated with T , is a factor of n_C .

2) Let G be a group of group of type G_2 over k arising from an octonion division algebra C . Let $K_0 = k(\sqrt{\delta}) \subset C$ be a fixed quadratic subfield. Note that $\langle 1, -\delta \rangle$ is a factor of n_C . Take $K = k \times k$ and $L = k \times K_0$. Let T be the K -unitary torus associated with the pair (L, K) . Since C is a division algebra, we have $K \not\subset C$. Hence, by Theorem 4.3, T does not embed in G over k , but the quadratic form $q_T = \langle 1, -\delta \rangle$ associated with T , is a factor of n_C .

Let A be an Albert algebra over k and $G = \mathbf{Aut}(A)$. Let L, K be étale algebras of dimension 3, 2 resp. and T be the K -unitary torus associated with the pair (L, K) . By case (2) of Remark 4.4, if there is a k -embedding $T \hookrightarrow G$, then K need not embed in $\text{Oct}(G)$, i.e. if $K = k(\sqrt{\alpha})$ then $\langle 1, -\alpha \rangle$ is not a factor of $f_3(G)$ in general. However,

Theorem 4.11. Let A be an Albert algebra over k and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale k -algebra and L be a cubic étale k -algebra. Let T be the K -unitary torus associated with the pair (L, K) . Suppose there exists a k -embedding $T \hookrightarrow G$. Then $f_5(A) = \langle 1, -\alpha \rangle \otimes \gamma$ for some 4-fold Pfister form γ over k .

Proof. We first assume that $K \cong k \times k$. If L is not a field, then $L = k \times K_0$, where K_0 is a quadratic étale k -algebra. By Lemma 4.1, $T \cong R_{K_0/k}(\mathbb{G}_m) \hookrightarrow G$. Therefore T is k -isotropic and k -rank of $G \geq 1$. Hence by Lemma 2.2, $f_5(A) = 0$. Let L be a field extension. Base changing to L we have, $L \otimes L \cong L \times K_0$, where $K_0 = L \otimes \Delta$ and Δ is the discriminant algebra of L over k . Hence by the above argument, $T \otimes L$ is L -isotropic and $f_5(A \otimes L) = 0$. By Springer's theorem $f_5(A) = 0$. Therefore, if $\alpha \in k^{*2}$ (i.e, $K = k \times k$) and $T \hookrightarrow G$ over k , then $f_5(A) = 0$. Using the same arguments as in Theorem 4.7, with Pfister form $\langle 1, -\alpha \rangle$ instead of q_T , we get the desired result. \square

Theorem 4.12. Let G be a simple, simply connected algebraic group defined over k . Let L be a cubic étale k -algebra with discriminant K_0 . Suppose there exists an k -embedding $\mathbf{L}^{(1)} \hookrightarrow G$. We then have:

- (a) if G is of type G_2 or A_2 then $\text{Oct}(G)$ splits.
- (b) if G is of type F_4 then $f_5(G) = 0$ and $K_0 \subset \text{Oct}(G)$.

Proof. Let L be as in the hypothesis and $K = k \times k$. Let (E, τ) and T be the K -unitary algebra and torus resp. associated with the pair (L, K) . We have $(E, \tau) \cong (L \times L, \varepsilon)$, where $\varepsilon(x, y) = (y, x)$ for all $(x, y) \in L \times L$. Hence $T \cong \mathbf{L}^{(1)}$. Therefore $T \hookrightarrow G$ over k .

(a) Let G be a simply connected, simple group of type G_2 or A_2 . Then, by Theorem 4.3, $K \subseteq \text{Oct}(G)$. Since K is split, $\text{Oct}(G)$ splits.

(b) Let G be a k -group of type F_4 . Then by Theorem 4.11, $f_5(A) = \langle 1, -\alpha \rangle \otimes \gamma$ for some 4-fold Pfister form γ over k where $K = k(\sqrt{\alpha})$. Since $K = k \times k$, we have $\alpha \in k^{*2}$ and hence $f_5(A) = 0$. If K_0 is split then L has trivial discriminant. Hence, by Theorem 4.2, $f_3(G) = 0$ and $\text{Oct}(G)$ splits. Therefore $K_0 \subset \text{Oct}(G)$. If K_0 is a field extension, base changing to K_0 we see that $L \otimes K_0$ is a cubic étale algebra over K_0 of trivial discriminant. Applying Theorem 4.2 to the K_0 -embedding $\mathbf{L}^{(1)} \otimes K_0 \hookrightarrow G \otimes K_0$, we get $f_3(G \otimes K_0) = f_3(G) \otimes K_0 = 0$. Hence $\text{Oct}(G)$ splits over K_0 and thus $K_0 \subseteq \text{Oct}(G)$. \square

Remark 4.13. Let G be a group of type F_4 over k . Let L be a cubic étale k -algebra. Suppose there exists a k -embedding $\mathbf{L}^{(1)} \hookrightarrow G$. Then $f_3(G)$ may not be zero. We use the construction as in the case (2) of Remark 4.4. Let C be an octonion division algebra. Let $\Gamma = \text{diag}(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Note that $\text{Oct}(G) = C$. Let $F \subseteq C$ be a quadratic subfield and $L = k \times F$. Let T be the F -unitary torus associated with the pair (L, F) . As in the case (2) of Remark 4.4, $T \hookrightarrow G$. Note that $T \cong \mathbf{SU}((k \times F) \otimes F, 1 \otimes \bar{}) \cong R_{K/k}(\mathbb{G}_m) \cong \mathbf{L}^{(1)}$ (Theorem 3.1, case (ii)). Hence $\mathbf{L}^{(1)} \hookrightarrow G$ but $f_3(G) \neq 0$.

4.1 Cohomology of unitary tori

In this section we will use some definitions and results from ([13], §29). Fix a separable closure k_{sep} of k and let $\Gamma = \text{Gal}(k_{\text{sep}}/k)$. Let G be an algebraic group defined over k and let $\rho : G \rightarrow \mathbf{GL}(W)$ be a representation for a finite dimensional k -vector space W . Fix an element $w \in W$ and identify W with a k -subspace of $W_{\text{sep}} = W \otimes k_{\text{sep}}$. An element $w' \in W_{\text{sep}}$ is called a twisted ρ -form of w if $w' = \rho_{\text{sep}}(g)(w)$ for some $g \in G(k_{\text{sep}})$, where $\rho_{\text{sep}} = \rho \otimes k_{\text{sep}}$. Let $\bar{A}(\rho, w)$ denote the groupoid whose objects are the twisted ρ -form of w and whose morphisms $w' \rightarrow w''$ are the elements $g \in G(k_{\text{sep}})$ such that $\rho_{\text{sep}}(g)(w') = w''$. Let $A(\rho, w)$ denote the groupoid whose objects are the twisted ρ -forms of w which lie in W and morphisms $w' \rightarrow w''$ are the elements $g \in G(k)$ such that $\rho(g)(w') = w''$. Let X denote the Γ -set of objects of $\bar{A}(\rho, w)$. Then X^Γ is the set of objects of $A(\rho, w)$. Also, the set of orbits of $G(k)$ in X^Γ is the set of isomorphism classes $\text{Isom}(A(\rho, w))$ of objects of $A(\rho, w)$. Let $w' \in A(\rho, w)$. By $[w'] \in \text{Isom}(A(\rho, w))$ we will denote the isomorphism class of w' . Let $\mathbf{Aut}_G(w)$ denote the stabilizer of w in G .

Proposition 4.1. ([13], Prop. 29.1) *If $H^1(k, G) = 0$, there is a natural bijection of pointed sets*

$$\text{Isom}(A(\rho, w)) \leftrightarrow H^1(k, \mathbf{Aut}_G(w))$$

which maps the isomorphism class of w to the base point of $H^1(k, \mathbf{Aut}_G(w))$.

The bijection between these sets is as follows: for $w' \in A(\rho, w)$, choose $g \in G(k_{\text{sep}})$ such that $\rho_{\text{sep}}(g)(w) = w'$. Define a 1-cocycle class $[\alpha_\sigma] \in H^1(k, \mathbf{Aut}_G(w))$ by $\alpha_\sigma = g^{-1}\sigma(g)$. Conversely let $[\alpha] \in H^1(k, \mathbf{Aut}_G(w))$. Since $H^1(k, G) = 0$, $\alpha = g^{-1}\sigma(g)$ for some $g \in G(k_{\text{sep}})$. The corresponding object in $A(\rho, w)$ is $\rho_{\text{sep}}(g)(w)$.

We now specialize and adapt some of the computations done in ([13], §29.17), to the case

of unitary algebras. Let k be a field and K be a quadratic étale k - algebra. Let $\bar{}$ denote the non-trivial k -automorphism of K . Let L be an étale algebra of dimension n over k and $E = L \otimes K$ be the associated K -unitary algebra with the involution $\tau = 1 \otimes \bar{}$.

We first calculate $H^1(k, \mathbf{SU}(E, \tau))$. Let $W = E \oplus K$. Define a representation $\rho : \mathbf{GL}_1(E) \longrightarrow \mathbf{GL}(W)$ by,

$$\rho(b)(x, y) = (bx\tau(b), N_{E \otimes k_s / K \otimes k_s}(b)y)$$

for all $b \in \mathbf{GL}_1(E)$, $x \in E$, $y \in K$. Let $w_0 = (1, 1) \in W$. Note that $\mathbf{GL}_1(E)(k_s) = (E \otimes k_s)^*$.

Claim: $\mathbf{Aut}_{\mathbf{GL}_1(E)}(w_0) = \mathbf{SU}(E, \tau)$.

We have,

$$\begin{aligned} \mathbf{Aut}_{\mathbf{GL}_1(E)}(w_0) &= \{g \in (E \otimes k_s)^* \mid \rho_{sep}(g)(1, 1) = (1, 1)\} \\ &= \{g \in (E \otimes k_s)^* \mid g\tau(g) = 1, N_{E \otimes k_s / K \otimes k_s}(g) = 1\} \\ &= \mathbf{SU}(E, \tau). \end{aligned}$$

Hence, in view of Proposition 4.1, we have a bijection

$$\eta : \text{Isom}(A(\rho, w_0)) \leftrightarrow H^1(k, \mathbf{SU}(E, \tau)).$$

We define a product on $\text{Isom}(A(\rho, w_0))$ as follows:

$$\rho_{sep}(g)(w_0)\rho_{sep}(g')(w_0) := \rho_{sep}(gg')(w_0) \text{ for all } g, g' \in (E \otimes k_{sep})^*.$$

A routine calculation shows that this product is well defined. Since $\mathbf{SU}(E, \tau)$ is a torus, $H^1(k, \mathbf{SU}(E, \tau))$ is an abelian group. It is immediate that η is a homomorphism of groups. Define

$$V := \{(s, z) \in L^* \times K^* \mid N_{L/k}(s) = z\bar{z}\}.$$

Given a twisted ρ -form w' of w_0 which lies in W , there exists $b \in (E \otimes k_{sep})^*$ such that $w' = \rho_{sep}(b)(w_0)$. Now $\rho_{sep}(b)(w_0) = \rho_{sep}(b)(1, 1) = (b\tau(b), N_{E \otimes k_{sep} / (K \otimes k_{sep})}(b))$. Along similar lines as in ([13], §29.17), we can show that $\rho_{sep}(b)(w_0) \in V$ and V is precisely the set of twisted ρ -forms of w_0 which lie in W . Define an equivalence \sim on V as follows:

$$(s, z) \sim (s', z') \text{ if and only if } s' = bs\tau(b) \text{ and } z' = N_{E/K}(b)z \text{ for some } b \in E^*.$$

We will denote equivalence class of $(s, z) \in V$ by $[(s, z)]$. Note that V is a subgroup of $L^* \times K^*$. It is easy to see that the product on V induces a well defined product on V / \sim as follows:

$$[(s, z)][(s', z')] = [(ss', zz')] \text{ for all } (s, z), (s', z') \in V.$$

Define $\xi : \text{Isom}(A(\rho, w_0)) \rightarrow V / \sim$ by $\xi([w']) = [(b\tau(b), N_{E \otimes k_{sep} / (K \otimes k_{sep})}(b))]$, where $w' = \rho_{sep}(b)(w_0)$, for some $b \in (E \otimes k_{sep})^*$. It follows that ξ is a homomorphism of groups.

We have proved the following,

Theorem 4.14. *Let L, K be étale k -algebras of dimension $n, 2$ resp. and T be the K -unitary torus associated with the pair (L, K) . Then there exists a natural isomorphism: $H^1(k, T) \mapsto V / \sim$ of groups.*

Henceforth we will identify $H^1(k, T)$ with V / \sim and write elements in $H^1(k, T)$ as equivalence classes $[(s, z)] \in V / \sim$.

Theorem 4.15. *Let L, K be étale k -algebras of dimension $n, 2$ resp. Let E be the K -unitary algebra and T be the K -unitary torus associated with the pair (L, K) . Then,*

$$\frac{H^1(k, T)}{K_0^{(1)}} \cong \frac{S}{N_{E/L}(E^*)},$$

where

$$S := \{u \in L^* \mid N_{L/k}(u) \in N_{K/k}(K^*)\} \text{ and } K_0^{(1)} := \{[(1, \mu)] \in H^1(k, T) \mid \mu\bar{\mu} = 1\}.$$

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \tau)$ is the K -unitary algebra associated with the pair (L, K) . Define $\phi : H^1(k, \mathbf{SU}(E, \tau)) \rightarrow \frac{S}{N_{E/L}(E^*)}$ by $\phi([(s, z)]) = sN_{E/L}(E^*)$. If $(s, z) \sim (s', z')$ then $s = s'b\tau(b)$ for some $b \in E$. Hence $s = s'N_{E/L}(E^*)$ and ϕ is well defined. We now check that ϕ is surjective. Let $s \in S$. By definition, there exists $z \in K^*$ such that $N_{L/k}(s) = z\bar{z}$, for some $z \in K$. Hence $\phi([(s, z)]) = s$, showing that ϕ is onto. Clearly ϕ is a homomorphism. Now, $\text{Ker } \phi = \{[(s, z)] \mid s \in N_{E/L}(E^*)\}$. Clearly, $K_0^{(1)} \subseteq \text{Ker } \phi$. Let $[(s, z)] \in \text{Ker } \phi$. Then $s \in N_{E/L}(E^*) = e\tau(e)$ for some $e \in E$. Let $\mu = zN_{E/K}(e^{-1})$. Then

$$N_{L/k}(s) = z\bar{z} = N_{E/K}(s) = N_{E/K}(e\tau(e)) = N_{E/K}(e)\overline{N_{E/K}(e)}.$$

Hence $z\bar{z} = N_{E/K}(e)\overline{N_{E/K}(e)}$. Therefore $\mu\bar{\mu} = 1$. It follows that $(s, z) = (e\tau(e), z) \sim (1, \mu)$. Hence, $\text{Ker } \phi = K_0^{(1)}$. \square

We obtain below an explicit expression for $H^1(k, \mathbf{SU}(E, \tau))$. Consider the exact sequence,

$$1 \longrightarrow K_0^{(1)} \xrightarrow{q} H^1(k, \mathbf{SU}(E, \tau)) \xrightarrow{\phi} \frac{S}{N_{E/L}(E^*)} \longrightarrow 1,$$

where q denotes the inclusion map and ϕ is as above. We provide a splitting of this sequence when dimension of L is odd. We will from here on assume that the k -dimension n of L is odd. Let $n = 2r + 1$.

Define $t : H^1(k, \mathbf{SU}(E, \tau)) \rightarrow K_0^{(1)}$ by,

$$t([(u, \mu)]) = [(1, \mu^{-r}\bar{\mu}^r)].$$

We first check that this map is well defined. Let $w \in E^*$. Then $(u, \mu) \sim (wu\tau(w), N_{E/K}(w)\mu)$. Now $w^{-r}\tau(w)^r \in U(E, \tau)$. Hence

$$(1, \mu^{-r}\bar{\mu}^r) \sim (1, N_{E/K}(w^{-r}\tau(w)^r)\mu^{-r}\bar{\mu}^r) = (1, (N_{E/K}(w)\mu)^{-r}\overline{(N_{E/K}(w)\mu)^r}).$$

Therefore, t is well defined. It is immediate that t is a homomorphism. We have

$$t \circ q([(1, \mu)]) = [(1, \mu^{-r}\bar{\mu}^r)] = [(1, \mu)]$$

(Since $\mu^{2r+1} = N_{E/K}(\mu)$ and $\mu\bar{\mu} = 1$). Hence, $t \circ q = \text{Id}_{K_0^{(1)}}$. Therefore there exists a homomorphism $\psi : \frac{S}{N_{E/L}(E^*)} \rightarrow H^1(k, \mathbf{SU}(E, \tau))$ such that $\phi \circ \psi = \text{Id}$. In fact ψ is given by,

$$\psi([u]) := [(u, \mu)]q(t([(u, \mu)]))^{-1} = [(u, \mu^{r+1}\bar{\mu}^{-r})],$$

where $N_{L/k}(u) = \mu\bar{\mu}$, $\mu \in K$. We now make some observations based on the above exact sequence. We have,

$$\begin{aligned} \text{Ker } t &= \{[(u, \mu)] \mid (1, \mu^{-r}\bar{\mu}^r) \sim (1, 1)\} = \{[(u, \mu)] \mid \mu^{-r}\bar{\mu}^r \in N_{E/K}(U(E, \tau))\}. \\ &= \text{Image } \psi = \{[(u, \mu^{r+1}\bar{\mu}^{-r})] \mid N_{L/k}(u) = \mu\bar{\mu}\}. \end{aligned}$$

Since ψ is an injective homomorphism, we have $\text{Image } \psi \cong \frac{S}{N_{E/L}(E^*)}$. Hence

$$\frac{S}{N_{E/L}(E^*)} \cong \{[(u, \mu)] \mid \mu^{-r}\bar{\mu}^r \in N_{E/K}(U(E, \tau))\} \cong \{[(u, \mu^{r+1}\bar{\mu}^{-r})] \mid N_{L/k}(u) = \mu\bar{\mu}\}.$$

Owing to the splitting of the exact sequence above we have, $H^1(k, \mathbf{SU}(E, \tau)) = \text{Image } q \times \text{Ker } t$. We have already seen that $\text{Ker } t \cong \frac{S}{N_{E/L}(E^*)}$. Now, $\text{Image } q = K_0^{(1)}$. Let $K^{(1)}$ denote the norm 1 elements of K . Define a map

$$\chi : K^{(1)} \longrightarrow K_0^{(1)}$$

by $\chi(\mu) := [(1, \mu)]$ for all $\mu \in K^{(1)}$. This map is clearly a surjective homomorphism. Now,

$$\begin{aligned} \text{Ker } \chi &= \{\mu \in K^{(1)} \mid [(1, \mu)] = [(1, 1)]\}. \\ &= \{\mu \in K^{(1)} \mid \mu = N_{E/K}(w), w\tau(w) = 1, w \in E\}. \\ &= N_{E/K}(U(E, \tau)). \end{aligned}$$

Hence, $\frac{K^{(1)}}{N_{E/K}(U(E, \tau))} \cong K_0^{(1)}$. We summarize this as:

Theorem 4.16. *Let K be a quadratic étale k -algebra and L be an étale k -algebra of dimension $n = 2r + 1$. Let E be the K -unitary algebra and T the K -unitary torus associated with the pair (L, K) . Then,*

$$H^1(k, T) \cong \frac{K^{(1)}}{N_{E/K}(U(E, \tau))} \times \frac{S}{N_{E/L}(E^*)}.$$

In fact, an explicit isomorphism is as follows:

$$\begin{aligned} \phi : H^1(k, T) &\longrightarrow K_0^{(1)} \times \frac{S}{N_{E/L}(E^*)} \\ \phi([(u, \mu)]) &= [(1, \mu^{-r}\bar{\mu}^r)], [u]. \end{aligned}$$

We now prove a somewhat analogous result to Theorem 4.15, for the cohomology of a unitary torus.

Theorem 4.17. *Let L, K be étale k -algebras of dimension $n, 2$ resp. Let E be the associated K -unitary algebra and T the K -unitary torus associated with the pair (L, K) . Then,*

$$\frac{H^1(k, T)}{L_0^{(1)}} \cong \frac{M}{N_{E/K}(E^*)},$$

where

$$M = \{\mu \in K^* \mid \mu\bar{\mu} \in N_{L/k}(L^*)\} \text{ and } L_0^{(1)} = \{[(u, 1)] \mid N_{L/k}(u) = 1\}.$$

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \tau)$ is the K -unitary algebra associated with the pair (L, K) . We define a map

$$\phi : H^1(k, \mathbf{SU}(E, \tau)) \longrightarrow \frac{M}{N_{E/K}(E^*)}$$

by $[(s, z)] \mapsto zN_{E/K}(E^*)$. It is easy to see that ϕ is a well defined surjective homomorphism and $\text{Ker } \phi = \{[(s, z)] \mid z \in N_{E/K}(E^*)\}$. Clearly, $L_0^{(1)} \subseteq \text{Ker } \phi$. Let $[(s, z)] \in \text{Ker } \phi$. Then $z = N_{E/K}(w)$, for some $w \in E^*$. Let $u = w^{-1}s\tau(w^{-1})$. Now, $N_{E/K}(u) = N_{E/K}(w^{-1}\tau(w^{-1})s)$ and

$$N_{E/K}(s) = N_{L/k}(s) = z\bar{z} = N_{E/K}(w\tau(w)).$$

Hence $N_{E/K}(u) = 1$. Also $(u, 1) = (w^{-1}s\tau(w^{-1}), 1) \sim (s, z)$. Therefore we have $\text{Ker } \phi = L_0^{(1)}$ \square

We now provide a decomposition of $H^1(k, \mathbf{SU}(E, \tau))$ analogous to that in Theorem 4.16. Consider the exact sequence,

$$1 \longrightarrow L_0^{(1)} \xrightarrow{q} H^1(k, \mathbf{SU}(E, \tau)) \xrightarrow{\phi} \frac{M}{N_{E/K}(E^*)} \longrightarrow 1,$$

where the maps q and ϕ are as above. We provide a splitting of this sequence when dimension of L is odd. We will from here on assume that the k -dimension n of L is odd. Let $n = 2r + 1$. We define a map

$$t : H^1(k, \mathbf{SU}(E, \tau)) \longrightarrow L_0^{(1)}$$

by $t([(u, \mu)]) := [(u^n N_{L/k}(u^{-1}), 1)]$. We first check that this map is well defined. Let $w \in E^*$. Then $(u, \mu) \sim (wu\tau(w), N_{E/K}(w)\mu)$. Since $N_{E/K}(w^n N_{E/K}(w^{-1})) = 1$ and $N_{E/K}(w^{-1}\tau(w^{-1})) = N_{L/k}(w^{-1}\tau(w^{-1}))$ we have,

$$\begin{aligned} (u^n N_{L/k}(u^{-1}), 1) &\sim (w^n N_{E/K}(w^{-1})u^n N_{L/k}(u^{-1})\tau(w^n)N_{E/K}(\tau(w^{-1})), 1) \\ &= ((wu\tau(w))^n N_{L/k}((wu\tau(w))^{-1}), 1). \end{aligned}$$

Therefore, t is well defined. It is easily checked that t is a homomorphism. Since $u \in L$ and $N_{E/K}(u) = N_{L/k}(u) = u\tau(u) = 1$, we have,

$$t \circ q([(u, 1)]) = [(u^{2r+1}, 1)] = [(u^{2r-1}(u\tau(u)), 1)] = [(u^{2r-1}, 1)].$$

By a similar calculation, $[(u^{2r-1}, 1)] = [(u^{2r-3}, 1)] = \dots = [(u, 1)]$. Hence $[(u^n, 1)] = [(u, 1)]$ and therefore $t \circ q = \text{Id}_{L_0^{(1)}}$.

Hence there exists a homomorphism $\psi : \frac{M}{N_{E/K}(E^*)} \longrightarrow H^1(k, \mathbf{SU}(E, \tau))$ such that $\phi \circ \psi = \text{Id}$. Explicitly, this map is given by

$$\psi([\mu]) := [(u, \mu)]q(t([(u^{-1}, \mu^{-1})])) = [(u^{-n+1}N_{L/k}(u), \mu)]$$

where $N_{L/k}(u) = \mu\bar{\mu}$, $\mu \in K^*$. We now make some observations based on this exact sequence. We have,

$$\begin{aligned} \text{Ker } t &= \{[(u, \mu)] [(u^n N_{L/k}(u^{-1}), 1)] \sim (1, 1)\} \\ &= \text{Image } \psi \\ &= \{[(u^{-n+1} N_{L/k}(u), \mu)] | N_{L/k}(u) = \mu\bar{\mu}\}. \end{aligned}$$

Since ψ is an injective group homomorphism, $\text{Image } \psi \cong \frac{M}{N_{E/K}(E^*)}$. Hence

$$\frac{M}{N_{E/K}(E^*)} \cong \{[(u, \mu)] [(u^n N_{L/k}(u^{-1}), 1)] \sim (1, 1)\} \cong \{[(u^{-n+1} N_{L/k}(u), \mu)] | N_{L/k}(u) = \mu\bar{\mu}\}.$$

Since the above sequence is split exact, we have $H^1(k, \mathbf{SU}(E, \tau)) = \text{Image } q \times \text{Ker } t$. We have already seen that $\text{Ker } t \cong \frac{M}{N_{E/K}(E^*)}$ and $\text{Image } q = L_0^{(1)}$. Let $L^{(1)}$ denote norm 1 elements of L and $E^{(1)} = \{x \in E | N_{E/K}(x) = 1\}$. Now define $\phi : L^{(1)} \rightarrow L_0^{(1)}$ by $u \mapsto [(u, 1)]$. It is easily checked that $\text{Ker } \phi = N_{E/L}(E^{(1)})$. Hence $\frac{L^{(1)}}{N_{E/L}(E^{(1)})} \cong L_0^{(1)}$. We summarize this as,

Theorem 4.18. *Let K be a quadratic étale k -algebra and L be an étale k -algebra of dimension $n = 2r + 1$. Let E be the K -unitary algebra and T the K -unitary torus associated to the pair (L, K) . Then,*

$$H^1(k, T) \cong \frac{L^{(1)}}{N_{E/L}(E^{(1)})} \times \frac{M}{N_{E/K}(E^*)}.$$

We now discuss the special case, when $L = k \times K_0$, where K_0 is a quadratic étale extension of k .

Theorem 4.19. *Let K and K_0 be a quadratic étale extensions of k . Let $L = k \times K_0$ and T be the K -unitary torus associated with the pair (L, K) . Then $H^1(k, T) = K_0^*/N_{K \otimes K_0/K_0}(K_0^*)$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$ is the K -unitary algebra associated with the pair (L, K) . Let $M := K \otimes K_0$. Then $L \otimes K \cong K \times (K \otimes K_0) = K \times M$. Now,

$$\begin{aligned} \mathbf{SU}(E, \tau) &\cong \{(a, x) \in K \times M | a\bar{a} = 1, x\tau(x) = 1, aN_{M/K}(x) = 1\} \\ &= \{(N_{M/K}(x^{-1}), x) | x\tau(x) = 1, x \in M^*\}. \end{aligned}$$

It follows that $\mathbf{SU}(E, \tau) \cong R_{K_0/k}^{(1)}(M)$. By Shapiro's Lemma ([13], Lemma 29.6),

$$H^1(k, R_{K_0/k}^{(1)}(M)) = H^1(K_0, \mathbf{M}^{(1)}). \text{ Hence, } H^1(k, T) = K_0^*/N_{K \otimes K_0/K_0}(K_0^*). \quad \square$$

Corollary 4.1. *Let $K = k(\sqrt{\alpha})$ and K_0 be quadratic field extensions of k . Let $L = k \times K_0$ and T be the K -unitary torus associated with the pair (L, K) . Then $H^1(k, T) = 0$ if and only if the quadratic form $\langle 1, -\alpha \rangle$ becomes universal over K_0 .*

Corollary 4.2. *Let K be a quadratic étale extension of k and $L = k \times K_0$, where K_0 is a quadratic étale extension of k . Let T be the K -unitary torus associated with the pair (L, K) . Let $H^1(k, T) = 0$. Then any composition algebra of dimension ≥ 4 which contains K contains K_0 .*

Proof. By Theorem 4.19, $H^1(k, T) = 0$ if and only if $N_{K \otimes K_0/K_0}(K \otimes K_0)^* = K_0^*$. Let C be a composition algebra properly containing K . Then $K \otimes K_0 \subseteq C \otimes K_0$. By doubling, $C \otimes K_0 = (K \otimes K_0) \oplus (K \otimes K_0).x$, for some $x \in (K \otimes K_0)^\perp$, $N_{C \otimes K_0}(x) \neq 0$. But since $K_0^* = N_{K \otimes K_0/K_0}(K \otimes K_0)^*$, we have $N_{C \otimes K_0}(x) \in N_{K \otimes K_0/K_0}(K \otimes K_0)^*$. Hence $C \otimes K_0$ is split and $K_0 \subseteq C$ ([5], Lemma 5). \square

One may be tempted to believe that for a distinguished K -unitary torus T , $H^1(k, T) = 0$. We give below an example to show that this is false. We also produce an example of a non-distinguished (k -anisotropic) torus T such that $H^1(k, T) = 0$.

Example 1: Let $k = \mathbb{R}(x)$ and $\delta = -1$. Choose $\alpha \in k^*$ such that $\alpha \notin k^{*2}$ and $\alpha \neq \delta \pmod{k^{*2}}$. Let $K = k(\sqrt{\alpha})$ and $K_0 = k(\sqrt{\delta})$. Then K_0 and K are fields. Also note that $K_0 = \mathbb{C}(x)$ is a C_1 field. Let $L = k \times K_0$, and T be the K -unitary torus associated with the pair (L, K) . Then, as in the proof of Theorem 4.19, $T \cong R_{K_0/k}^{(1)}(K \otimes K_0)$. Also by Theorem 4.19, $H^1(k, T) = K_0^*/N_{K \otimes K_0/K_0}(K \otimes K_0)$. Since $\alpha \neq \delta \pmod{k^{*2}}$, T is not distinguished. By Corollary 4.1, $H^1(k, T) = 0$ if and only if the binary form $< 1, -\alpha >$ becomes universal over K_0 . Since K_0 is a C_1 field, all binary forms over K_0 are universal, in particular $< 1, -\alpha >$ becomes universal over K_0 . Hence $H^1(k, T) = 0$. Note that since $\alpha \neq \delta \pmod{k^{*2}}$, $K \otimes K_0$ is a field. Hence by ([23], Example on Pg. 54), T is k -anisotropic. This also gives an example of a k -anisotropic K -unitary torus T such that $H^1(k, T) = 0$.

Example 2: Let $K = k \times k$ and L be a cyclic cubic field extension of k . Let T be the K -unitary torus associated with the pair (L, K) . By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$. Note that $(E, \tau) = (L \times L, \epsilon)$. Hence $T \cong \mathbf{L}^{(1)}$. Now

$$H^1(k, T) \cong H^1(k, \mathbf{L}^{(1)}) \cong \frac{k^*}{N_{L/k}(L^*)}.$$

Let $p(X) = X^3 - 3X - 1 \in \mathbb{Q}[X]$, then $p(X)$ is irreducible over \mathbb{Q} . Let $L' := \mathbb{Q}[X]/<p(X)>$. Then L' is a cyclic cubic extension of \mathbb{Q} such that $N_{L'/\mathbb{Q}}(L'^*) \neq \mathbb{Q}^*$ ([29], Pg. 186). Let T be the K -unitary torus associated with the pair (L', K) . Then T is a distinguished torus such that $H^1(\mathbb{Q}, T) \neq 0$.

Example 3: Let T be a distinguished k -torus arising from a pair (L, K) where L is a cubic étale k -algebra which is not a field extension and K is a quadratic étale k -algebra. By Theorem 3.1, T is either $\mathbb{G}_m \times \mathbb{G}_m$ or $R_{K/k}(\mathbb{G}_m)$. In either case, by Hilbert theorem 90 and Shapiro's Lemma ([13], Lemma 29.6), $H^1(k, T) = 0$.

Example 4: Let $L = k \times k \times k$ and K be a quadratic étale extension of k . Then

$$\mathbf{SU}(E, \tau) = \{(x, y, z) \in K \times K \times K \mid x\bar{x} = y\bar{y} = z\bar{z} = 1, xyz = 1\} \cong K^{(1)} \times K^{(1)}.$$

Thus $\mathbf{SU}(E, \tau) \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$ and $H^1(k, \mathbf{SU}(E, \tau)) = k^*/N_{K/k}(K^*) \times k^*/N_{K/k}(K^*)$. Hence $H^1(k, \mathbf{SU}(E, \tau)) = 0$ if and only if $k^* = N_{K/k}(K^*)$. Let A be an Albert algebra. Let T be the K -unitary torus associated with the pair (L, K) such that $H^1(k, T) = 0$. Let $T \hookrightarrow \mathbf{Aut}(A)$ over k . Since L has trivial discriminant over k , by Theorem 4.5, $K \subset \mathbf{Oct}(A)$. Since $H^1(k, T) = 0$, $k^* = N_{K/k}(K^*)$. Hence $f_3(A) = 0$.

Theorem 4.20. *Let K be a quadratic étale extension of k and $L = k \times K_0$, where K_0 is a quadratic étale algebra over k . Let T be the K -unitary torus associated with the pair (L, K) with $H^1(k, T) = 0$. Let A be an Albert algebra over k . If there exists a k -embedding $T \hookrightarrow \mathbf{Aut}(A)$, then $K_0 \subset \text{Oct}(A)$.*

Proof. If $K_0 = k \times k$, then by Example 4, $\text{Oct}(A)$ splits. Hence $K_0 \subset \text{Oct}(A)$. Let K_0 be a field extension. Base changing to K_0 we have,

$$T \otimes K_0 = \mathbf{SU}((L \otimes K_0) \otimes_{K_0} (K_0 \otimes K), \tau \otimes 1) \hookrightarrow \mathbf{Aut}(A) \otimes K_0.$$

Since $L \otimes K_0$ has trivial discriminant over K_0 , by Theorem 4.5, $K_0 \otimes K \subset \text{Oct}(A) \otimes K_0$. By doubling, $\text{Oct}(A) \otimes K_0 = (K \otimes K_0) \oplus (K \otimes K_0).x$, for some $x \in (K \otimes K_0)^\perp$, $N_{\text{Oct}(A) \otimes K_0}(x) \neq 0$. But since $H^1(k, T) = 0$, by Theorem 4.19 we have, $K_0^* = N_{K \otimes K_0/K_0}(K \otimes K_0)^*$. Hence $N_{\text{Oct}(A) \otimes K_0}(x) \in N_{K \otimes K_0/K_0}(K \otimes K_0)^*$. Therefore $\text{Oct}(A) \otimes K_0$ is split and by ([5], Lemma 5), $K_0 \subseteq \text{Oct}(A)$. \square

5 Étale Tits processes of Jordan algebras and applications

In this section, we develop some results on étale Tits processes, in the context of unitary tori. Let L be a cubic étale algebra and K be a quadratic étale algebra of an arbitrary base field k and $E = L \otimes K$. Let $\tau = 1 \otimes \bar{}$, where $\bar{}$ denotes the non-trivial involution on K . Suppose $(u, \mu) \in L^* \times K^*$ is such that $N_{L/k}(u) = N_{K/k}(\mu)$. We call the pair (u, μ) an **admissible pair**. The étale Tits process produces an absolutely simple Jordan algebra $J = J(E, \tau, u, \mu)$ of degree 3 and dimension 9, with the underlying vector space $L \oplus E$ and with $L = \{(l, 0) \mid l \in L\}$ as a subalgebra. Let (B, σ) be a central simple algebra over K with an involution σ of the second kind and let $(B, \sigma)_+$ contain a cyclic étale algebra L over k . Then $B = L \otimes K \oplus (L \otimes K)z \oplus (L \otimes K)z^2$ with $z^3 = \mu \in K^*$ such that $N_{K/k}(\mu) = 1$. Also the involution σ is given by $\sigma(z) = uz^{-1}$ with $u \in L$ such that $N_B(u) = 1$. In this case $(B, \sigma)_+ \cong J(L \otimes K, \tau, u, \mu)$ (see [13], Pg. 527 for details). We define étale Tits processes J_1 and J_2 arising from étale algebras L and K of dimensions 3, 2 resp., to be L -isomorphic, denoted by $J_1 \cong_L J_2$, if there exists a k -isomorphism $J_1 \rightarrow J_2$, which restricts to the subalgebra L of J_1 and J_2 . By ([18], Prop. 3.7) we have the following

Theorem 5.1. ([18], Prop. 3.7) *Let L, K and E be as above. Let (u, μ) be an admissible pair. For any $w \in E$, $(wu\tau(w), \mu N_{E/K}(w))$ is again an admissible pair and*

$$J(E, \tau, u, \mu) \cong_L J(E, \tau, wu\tau(w), \mu N_{E/K}(w)),$$

via $(a, b) \mapsto (a, bw)$.

Remark 5.2. Note that $J(E, \tau, 1, 1)$ has zero divisors. Choose $x \in E$ such that $\tau(x) = -x$. Then $(0, x)$ is a zero divisor in $J(E, \tau, 1, 1)$, since it is a norm zero element. More generally, as an easy consequence of Theorem 5.1, one can see that if $\mu \in N_{E/K}(E^*)$, then $J(E, \tau, u, \mu)$ has zero divisors.

Theorem 5.3. *For any étale Tits construction $J(E, \tau, u, \mu)$, there exists an L -isomorphic Tits process $J(E, \tau, u', \mu')$ with $N_{L/k}(u') = 1 = \mu'\tau(\mu')$.*

Proof. Take $w = \mu^{-1}u$ and apply Theorem 5.1. \square

Lemma 5.1. *Let L, K be étale k -algebras of dimension 3, 2 resp. Let (E, τ) be the K -unitary algebra associated with the pair (L, K) . Suppose $\phi : J(E, \tau, u, \mu) \rightarrow J(E, \tau, v, \nu)$ is an L -isomorphism. Then there exists $w \in E$ such that $u = \phi^{-1}(v)w\tau(w)$ and $\mu = N_{E/K}(w)\nu$ or $\mu = N_{E/K}(w)\bar{\nu}$.*

Proof. By definition, $(E, \tau) = (L \otimes K, 1 \otimes \bar{})$. By Theorem 5.3, we may assume $N_{L/k}(v) = v\bar{v} = 1$. Let $\phi : J(E, \tau, u, \mu) \rightarrow J(E, \tau, v, \nu)$ be an L -isomorphism. Define $h : L \oplus E \rightarrow L$ and $g : L \oplus E \rightarrow E$ by

$$\phi(a, b) = (h(a, b), g(a, b)),$$

for $a \in L, b \in E$. Since ϕ is an isomorphism, one can easily check that g and h are k -linear maps. Since ϕ is an isomorphism of Jordan algebras, it preserves the trace forms on both the algebras. Note that L^\perp in $J(E, \tau, u, \mu)$ with respect to the trace form, is the k -subspace $\{(0, e) \mid e \in E\}$, and similarly for $J(E, \tau, v, \nu)$. Since ϕ restricts to L , ϕ maps L^\perp in $J(E, \tau, u, \mu)$ to L^\perp in $J(E, \tau, v, \nu)$. Hence for $b \in E$, $\phi(0, b) = (0, b')$ for some $b' \in E$. It follows that $h(0, b) = 0$ for all $b \in E$. Therefore $h(a, b) = h(a, 0)$ for all $a \in L, b \in E$. We will now on write simply $h(a)$ for $h(a, b)$. Since ϕ is an isomorphism of Jordan algebras, it is easy to check that $h : L \rightarrow L$ is an automorphism. Since ϕ restricts to L , $\phi(a, 0) = (h(a), 0)$ for all $a \in L$. Hence $g(a, 0) = 0$ for all $a \in L$. It follows that $g(a, b) = g(0, b)$ for all $a \in L, b \in E$. We will now on write simply $g(b)$ for $g(0, b)$. Again since ϕ is an isomorphism of Jordan algebras, it is easy to check that $g : E \rightarrow E$ is a bijection. Since ϕ preserves norms, $N(a, b) = N(h(a), g(b))$. Expanding norms we get,

$$\begin{aligned} & N_{L/k}(a) + \mu N_{E/k}(b) + \overline{\mu N_{E/k}(b)} - t_{L/k}(abu\tau(b)) \\ &= N_{L/k}(h(a)) + \nu N_{E/k}(g(b)) + \overline{\nu N_{E/k}(g(b))} - t_{L/k}(h(a)g(b)\nu\tau(g(b))). \end{aligned}$$

Putting $a = 0$, we get

$$\mu N_{E/k}(b) + \overline{\mu N_{E/k}(b)} = \nu N_{E/k}(g(b)) + \overline{\nu N_{E/k}(g(b))}, \quad b \in E.$$

Since h is an automorphism of L , we have $N_{L/k}(a) = N_{L/k}(h(a))$, $a \in L$. Hence we get,

$$t_{L/k}(abu\tau(b)) = t_{L/k}(h(a)g(b)\nu\tau(g(b))), \quad a \in L, b \in E.$$

Putting $b = 1$, we get,

$$t_{L/k}(au) = t_{L/k}(h(a)g(1)\nu\tau(g(1))), \quad a \in L.$$

Since $g(1)\nu\tau(g(1)) \in L$, there exist $b \in L$ such that $h(b) = g(1)\nu\tau(g(1))$. Hence

$$t_{L/k}(au) = t_{L/k}(h(a)h(b)) = t_{L/k}(h(ab)) = t_{L/k}(ab)$$

for all $a \in L$. Since the trace bilinear form $T(a, b) := T_{L/k}(ab)$ on L/k is non-degenerate, we have $u = b$. Let $\hat{h} : E \rightarrow E$ be defined by $\hat{h} = h \otimes 1$. Then \hat{h} is the extension of h to a K -automorphism of E . In particular, \hat{h} commutes with τ . We have,

$$u = \hat{h}^{-1}(g(1))\hat{h}^{-1}(v)\hat{h}^{-1}\tau(g(1)) = \hat{h}^{-1}(g(1))h^{-1}(v)\tau(\hat{h}^{-1}(g(1))).$$

Hence $u = wh^{-1}(v)\tau(w) = \phi^{-1}(v)w\tau(w)$, where $w = \hat{h}^{-1}(g(1)) \in E$. This proves the first assertion in the Lemma.

Now we prove the assertion on μ . Let $h^{-1}(v) = v_0 \in L$. Then $N_{L/k}(v_0) = N_{L/k}(v) = 1$. Let $u_1, u_2 \in E$. Define,

$$\langle u_1 \rangle \cong \langle u_2 \rangle \text{ over } E \text{ if and only if } u_1 = wu_2\tau(w), \text{ for some } w \in E.$$

Hence, $\langle u \rangle \cong \langle v_0 \rangle$ over E .

We now introduce an equivalence on the set of admissible pairs in $L^* \times K^*$ as follows:

$(u_1, \mu_1) \sim (u_2, \mu_2)$ if and only if there exists $w \in E$ such that $u_2 = wu_1\tau(w)$ and $\mu_2 = N_{E/K}(w)\mu_1$ or $\mu_2 = N_{E/K}(w)\overline{\mu_1}$.

Claim $(u, \mu) \sim (v_0, v)$.

Since $J(E, \tau, u, \mu) \cong J(E, \tau, v, v)$, we have $J(E, \mu) \cong J(E, v)$ over K and by ([17], Prop. 4.3), $\mu \in vN_{E/K}(E^*)$ or $\mu \in \overline{v}N_{E/K}(E^*)$. Let $\mu = vN_{E/K}(w)$ or $\mu = \overline{v}N_{E/K}(w)$ as is the case accordingly, for some $w \in E$. Let $v' = w^{-1}u\tau(w^{-1})$. Then $N_{L/k}(v') = v\overline{v} = 1$ and $(v', v) \sim (u, \mu)$. Now $\langle u \rangle \cong \langle v' \rangle$ and $\langle u \rangle \cong \langle v_0 \rangle$ over E . Hence $\langle v' \rangle \cong \langle v_0 \rangle$ over E .

Therefore, there exists $w'' \in E$ such that $v_0 = w''v'\tau(w'')$. Let $\lambda = N_{E/K}(w'')$. Since $N_{L/k}(v_0) = N_{L/k}(v') = 1$, we have $\lambda\overline{\lambda} = 1$. Hence by ([17], Lemma 4.5), there exists $w_1 \in E$ such that $\lambda = N_{E/K}(w_1)$ and $w_1\tau(w_1) = 1$.

Therefore,

$$(v', v) \sim (v_0, \lambda v) \sim (v_0, v).$$

Thus $(u, \mu) \sim (v_0, v) = (u, \mu) \sim (h^{-1}(v), v) = (\phi^{-1}(v), v)$. Hence, by the definition of the equivalence, $\mu = N_{E/K}(w)v$ or $\mu = N_{E/K}(w)\overline{v}$. This completes the proof. \square

Remark 5.4. As a converse to the above lemma, if there exists $w \in E^*$ such that $u = \phi^{-1}(v)w\tau(w)$ and $\mu = N_{E/K}(w)v$ (or $\mu = N_{E/K}(w)\overline{v}$), where $\phi \in \text{Gal}(L/k)$, then $J(E, \tau, u, \mu)$ is L -isomorphic to $J(E, \tau, v, v)$ (resp. $J(E, \tau, v, \overline{v})$). To see this, suppose $\mu = N_{E/K}(w)v$. By Theorem 5.1,

$$J(E, \tau, u, \mu) = J(E, \tau, \phi^{-1}(v)w\tau(w), N_{E/K}(w)v) \cong_L J(E, \tau, \phi^{-1}(v), v).$$

Extend ϕ to an automorphism $\hat{\phi}$ of E , defined as $\hat{\phi} = \phi \otimes 1$. Note that $\hat{\phi}$ commutes with τ . Consider the map $\psi : J(E, \tau, \phi^{-1}(v), v) \rightarrow J(E, \tau, v, v)$ given by $(a, x) \mapsto (\phi(a), \hat{\phi}(x))$ for $a \in L, x \in E$. Clearly $\psi((1, 0)) = (1, 0)$. We have,

$$\begin{aligned} N(\phi(a), \hat{\phi}(x)) &= N_{L/k}(\phi(a)) + \mu N_{E/k}(\hat{\phi}(x)) + \overline{\mu N_{E/k}(\hat{\phi}(x))} - t_{L/k}(\phi(a)v\hat{\phi}(x)\tau(\hat{\phi}(x))) \\ &= N_{L/k}(a) + \mu N_{E/k}(x) + \overline{\mu N_{E/k}(x)} - t_{L/k}(\hat{\phi}(a\phi^{-1}(v)x\tau(x))) \\ &= N_{L/k}(a) + \mu N_{E/k}(x) + \overline{\mu N_{E/k}(x)} - t_{L/k}(a\phi^{-1}(v)x\tau(x)) = N(a, x). \end{aligned}$$

Hence ψ is a k -linear bijection preserving norms and identities. Therefore, by Theorem 2.1, ψ is an isomorphism of Jordan algebras. Also ψ restricts to L . Hence ψ is an L -isomorphism. When $\mu = N_{E/K}(w)\overline{v}$, a similar argument completes the proof.

Corollary 5.1. Let L, K be étale k -algebras of dimension 3, 2 resp. Let (E, τ) be the K -unitary torus associated with the pair (L, K) . There exists an L -isomorphism $J(E, \tau, u, \mu) \cong J(E, \tau, 1, 1)$ if and only if there exists $w \in E$ such that $u = w\tau(w)$ and $\mu = N_{E/K}(w)$.

Theorem 5.5. *There exists a surjective map from $H^1(k, \mathbf{SU}(E, \tau))$ to the set of L -isomorphism classes of étale Tits process algebras arising from (L, K) .*

Proof. Let X denote the set of L -isomorphism classes of étale Tits process algebras arising from (L, K) . Given an étale Tits process J , let $[J]$ denote the L -isomorphism class of J . Let $\phi : H^1(k, \mathbf{SU}(E, \tau)) \rightarrow X$ be defined by $\phi([(u, \mu)]) := [J(E, \tau, u, \mu)]$. Let $[(u, \mu)] \in H^1(k, \mathbf{SU}(E, \tau))$ and $[(u, \mu)] = [(v, \nu)]$. Then $u = v\tau(w)$ and $\mu = N_{E/K}(w)\nu$ for some $w \in E$. Hence by Theorem 5.1, $J(E, \tau, u, \mu)$ is L -isomorphic to $J(E, \tau, v, \nu)$. Therefore ϕ is well defined. Clearly ϕ is onto. \square

As an easy consequence of the above theorem, we note that if $H^1(k, \mathbf{SU}(E, \tau)) = 0$, then all étale Tits process algebras arising from (L, K) are isomorphic. More precisely, we have,

Theorem 5.6. *Let L, K be a étale k -algebras of dimension 3, 2 resp. and (E, τ) be the K -unitary algebra and T the K -unitary torus associated with the pair (L, K) . Then $H^1(k, T) = 0$ if and only if $J(E, \tau, u, \mu) \cong_L J(E, \tau, 1, 1)$, for all admissible pairs $(u, \mu) \in L^* \times K^*$.*

Proof. Suppose $J(E, \tau, u, \mu) \cong_L J(E, \tau, 1, 1)$ for all admissible pairs $(u, \mu) \in L^* \times K^*$. Let S be as in Theorems 4.15.

Claim: $S = N_{E/L}(E^*)$ and $K^{(1)} = N_{E/K}(U(E, \tau))$.

Let $u \in S$. Since $J(E, \tau, u, \mu) \cong_L J(E, \tau, 1, 1)$, by Corollary 5.1, $u = N_{E/L}(w) = w\tau(w)$ and $\mu = N_{E/K}(w)$ for some $w \in E$. Hence $S = N_{E/L}(E^*)$. Let $\mu_0 \in K^{(1)}$. Since $J(E, \tau, 1, \mu_0) \cong_L J(E, \tau, 1, 1)$, by Corollary 5.1, $\mu_0 = N_{E/K}(w)$ where $w\tau(w) = 1$.

Hence $K^{(1)} = N_{E/K}(U(E, \tau))$, and by Theorem 4.16, $H^1(k, T) = 0$. The converse follows immediately from Theorem 5.5. \square

Corollary 5.2. *Let L be a cubic étale k -algebra with discriminant δ and $K = k(\sqrt{\alpha})$ be a quadratic étale k -algebra. Let T be the K -unitary torus associated with the pair (L, K) and $H^1(k, T) = 0$. Let B be any degree 3 central simple algebra over $k(\sqrt{\alpha\delta})$ with an involution σ of the second kind such that $L \subseteq (B, \sigma)_+$. Then $B \cong M_3(k(\sqrt{\alpha\delta}))$ and σ is distinguished.*

Proof. By ([17], Theorem 1.6), there exists an admissible pair $(u, \mu) \in L^* \times K^*$ such that $\phi : (B, \sigma)_+ \cong J(E, \tau, u, \mu)$, where the isomorphism ϕ restricts to the identity of L . Since $H^1(k, T) = 0$, by Theorem 5.6, $(B, \sigma)_+ \cong J(E, \tau, u, \mu) \cong_L J(E, \tau, 1, 1)$. Since $J(E, \tau, 1, 1)$ is reduced (see Remark 5.2), by ([22], Theorem 1, cf. [17], Theorem 1.4), $B \cong M_3(k(\sqrt{\alpha\delta}))$. By Lemma 5.2, there exists $v \in L$ such that $\text{Int}(v) \circ \sigma$ is distinguished. Since ϕ restricts to the identity of L , taking isotopes with respect to v on both sides, we have $(B, \text{Int}(v) \circ \sigma)_+ \cong J(E, \tau, u, \mu)^{(v)}$ (see [18], Prop. 3.9). By ([18], Prop. 3.9) $J(E, \tau, u, \mu)^{(v)} \cong J(E, \tau, uv^\#, N(v)\mu) \cong J(E, \tau, 1, 1)$. Hence $(B, \sigma)_+ \cong (B, \text{Int}(v) \circ \sigma)_+$. By ([13], Prop. 37.6), we have $f_3((B, \sigma)) = f_3((B, \text{Int}(v) \circ \sigma)) = 0$ and σ is distinguished. \square

Corollary 5.3. *Let the hypothesis be as in Corollary 5.2. Let B be any degree 3 central simple algebra over $k(\sqrt{\alpha\delta})$ with an involution of the second kind such that $L \subseteq B$. Then*

$$B \cong M_3(k(\sqrt{\alpha\delta})).$$

Proof. By ([6], Prop.17, cf. [13], Cor. 19.30), there exists an involution σ on B such that $L \subseteq (B, \sigma)_+$. Hence by the above corollary B splits. \square

In view of Corollary 5.3, when L is a cubic étale algebra over k with trivial discriminant, we have the following

Corollary 5.4. *Let L be a cubic étale algebra over k with trivial discriminant and K be a quadratic étale k -algebra. Let T be the K -unitary torus associated with the pair (L, K) and $H^1(k, T) = 0$. Let B be any degree 3 central simple algebra over K with an involution σ of the second kind such that $L \subseteq B$. Then $B \cong M_3(K)$.*

Proof. Let L be as in the hypothesis. When $L = k \times k \times k$, it is immediate that $B \cong M_3(K)$. When L is a cubic cyclic field extension of k , by Corollary 5.3, we get the desired result. \square

5.1 Applications to groups of type A_2 and G_2

We now give some applications based on the cohomology computation of tori done in section §4.1. We shall consider cohomology of maximal tori in groups of type A_2 and G_2 . These tori arise from six dimensional unitary algebras, hence we can compute their cohomology using Theorems 4.18, 4.16 with $n = 3$. We deduce that a group of type G_2 splits if and only if it contains a maximal torus whose first cohomology vanishes. A weaker result holds for groups of type A_2 . We need a variant of ([6], Prop. 17) for our purpose.

Lemma 5.2. *Let $F = k(\sqrt{\alpha})$ be a quadratic étale k -algebra and B be a degree 3 central simple algebra over F with an involution σ of the second kind. Let L be a cubic étale algebra such that $L \subseteq (B, \sigma)_+$. Then there exists $l \in L$ with $N_{L/k}(l) \in k^{*2}$ such that $\text{Int}(l) \circ \sigma$ is distinguished.*

Proof. Since $L \subseteq (B, \sigma)_+$, by ([6], Proposition 11), there exists $\mu \in L^*$ with $N_{L/k}(\mu) \in k^{*2}$ such that

$$Q_\sigma = \langle 1, 2, 2\delta \rangle \perp \langle 2 \rangle \cdot \langle \alpha \delta \rangle \cdot t_{L/k}(\langle \mu \rangle).$$

Let $\lambda_0 \in L^*$ be such that $t_{L/k}(\lambda_0) = 0$ and let $\lambda := \frac{\lambda_0}{N_{L/k}(\lambda_0)}$. Then $\lambda \in L^*$ and $N_{L/k}(\lambda) \in k^{*2}$. Hence there exists $\xi \in k^*$ such that $N_{L/k}(\lambda\mu^{-1}) = \xi^2$. Consider $\psi := \text{Int}(\xi\lambda^{-1}\mu) \circ \sigma = \text{Int}(\lambda^{-1}\mu) \circ \sigma$.

Claim: ψ is a distinguished involution.

We will use the proofs of ([6] Prop. 17, Corollary 14). Since $\lambda^{-1}\mu \in L$, we have $L \subseteq (B, \psi)_+$. Let $q : L \rightarrow L$ be defined by, $lq(l) = n_{L/k}(l)$. By ([6], Proposition 13), we have

$$Q_\psi = \langle 1, 2, 2\delta \rangle \perp \langle 2 \rangle \cdot \langle \alpha \delta \rangle \cdot t_{L/k}(\langle q(\xi\lambda^{-1}\mu) \rangle).$$

It is easy to check that $q(\xi\lambda^{-1}\mu) = \lambda\mu^{-1}$. Hence

$$\begin{aligned} Q_\psi &= \langle 1, 2, 2\delta \rangle \perp \langle 2 \rangle \cdot \langle \alpha \delta \rangle \cdot t_{L/k}(\langle \lambda \rangle) \\ &= \langle 1, 1, 1 \rangle \perp \langle 2\delta \rangle \cdot \langle \alpha \rangle \cdot t_{L/k}(\langle \lambda \rangle). \end{aligned}$$

Let $(B, \sigma)_+^\circ = \{x \in (B, \sigma)_+ \mid T_B(x) = 0\}$ and Q_ψ° denote the restriction of Q_ψ to $(B, \sigma)_+^\circ$. Then $Q_\psi^\circ = \langle 2 \rangle \cdot \langle 1, 3 \rangle \perp \langle \delta \rangle \cdot \langle \alpha \rangle \cdot t_{L/k}(\langle \lambda \rangle)$. Since $t_{L/k}(\lambda) = 0$, the form $t_{L/k}(\langle \lambda \rangle)$ is isotropic over k and the Witt index of $\langle \alpha \rangle \cdot t_{L/k}(\langle \lambda \rangle)$ is at least two. Hence by ([13], Theorem 16 (c)), ψ is distinguished. \square

Theorem 5.7. *Let $F = k(\sqrt{\alpha})$ be a quadratic étale k -algebra and B be a degree 3 central simple algebra over F with an involution σ of the second kind. Let T be a maximal torus of $\mathbf{SU}(B, \sigma)$. If $H^1(k, T) = 0$ then σ is distinguished.*

Proof. Let T be a maximal torus of $\mathbf{SU}(B, \sigma)$. By Theorem 2.4, $T \cong \mathbf{SU}(E, \sigma)$, where $(E, \sigma) \subseteq (B, \sigma)$ is an F -unitary algebra. Let $L = E^\sigma$ and $\text{Disc}(L) = \delta$. By Lemma 2.3, $E = L \otimes F$. By Lemma 5.2, there exists $l \in L$, $N_{L/k}(l) \in k^{*2}$ such that $\text{Int}(l) \circ \sigma$ is distinguished. Let $\psi := \text{Int}(l) \circ \sigma$ and $S = \{u \in L^* \mid N_{L/k}(u) \in N_{F/k}(F^*)\}$. Since $H^1(k, T) = 0$, by Theorem 4.16 $\frac{S}{N_{E/L}(E^*)} = \{1\}$. Let $u \in S$. Then $u = w\sigma(w)$ for some $w \in E$ and $N_{L/k}(u) = \bar{\gamma}\gamma$ for some $\gamma \in F$. Consider the Albert algebra $A := J(B, \sigma, u, \gamma)$. By ([13] Lemma 39.2), $J(B, \sigma, u, \gamma) \cong J(B, \sigma, w'u\sigma(w'), N_B(w')\gamma)$ for all $w' \in B^*$. Hence for $w' = w^{-1}$, we have $A = J(B, \sigma, u, \gamma) \cong J(B, \sigma, w\sigma(w), \gamma) \cong J(B, \sigma, 1, \rho)$, where $\rho = N_B(w)^{-1}\gamma$. Therefore, $f_3(A) = f_3(B, \text{Int}(u) \circ \sigma) = f_3(B, \sigma)$ for all $u \in S$. Taking $u = l \in S$, we get $f_3(A) = f_3(B, \sigma) = 0$. Hence σ is distinguished. \square

Remark 5.8. A converse of the above theorem holds when B is split. Let $F = k(\sqrt{\alpha})$ be a quadratic étale k -algebra and σ be a distinguished involution on $M_3(F)$. Let $L = k \times F$. Note that $L \hookrightarrow M_3(F)$ as a k -subalgebra (via the embedding $(\gamma, x) \rightarrow \text{diag}(\gamma, x, x)$, $\gamma \in k$, $x \in F$). Since σ is distinguished, by ([6], Cor 18), there exists a k -embedding $L \hookrightarrow (M_3(F), \sigma)_+$. Let T be the F -unitary torus associated with the pair (L, F) . By Lemma 2.5, $T \hookrightarrow G$ over k . Then by case (ii) of the proof of Theorem 3.1, $T \cong R_{F/k}(\mathbb{G}_m)$. Hence $T \hookrightarrow \mathbf{SU}(B, \sigma)$ is a maximal k -torus with $H^1(k, T) = 0$.

Corollary 5.5. *Let L be a cubic étale algebra over k of trivial discriminant and K be a quadratic étale k -algebra. Let T be the K -unitary torus associated with the pair (L, K) and $H^1(k, T) = 0$. Let B be any degree 3 central simple algebra over K with an involution σ of the second kind such that $L \subseteq (B, \sigma)_+$. Then $B \cong M_3(K)$ and σ is distinguished.*

Proof. Since $T \hookrightarrow \mathbf{SU}(B, \sigma)$ over k (see Lemma 2.5) and $H^1(k, T) = 0$, by Theorem 5.7, σ is distinguished. Also, by Corollary 5.4, $B \cong M_3(K)$. \square

Theorem 5.9. *Let G be a group of type G_2 over k . Then G splits over k if and only if there exist a maximal k -torus $T \subset G$ such that $H^1(k, T) = 0$.*

Proof. Let $T \subset G$ be a maximal k -torus such that $H^1(k, T) = 0$. As in §2.8, there exists a quadratic étale k -algebra K and $h \in GL_3(k)$ such that $T \subseteq \mathbf{SU}(M_3(K), *_h) \subseteq G$, where $*_h$ denotes the involution on $M_3(K)$ given by $*_h(X) = h^{-1}\bar{X}^t h$. Since $H^1(k, T) = 0$, $*_h$ is a distinguished involution (see Theorem 5.7). Hence by ([8], Theorem 4.4), G splits over k . For the converse, we choose T to be a split maximal k -torus in G , then $H^1(k, T) = 0$. \square

The Real Case

Let G be a group of type F_4 over \mathbb{R} . Let L, K be étale algebras over \mathbb{R} of dimension 3, 2 resp. and T be the K -unitary torus associated with the pair (L, K) . Suppose $T \hookrightarrow G$ over \mathbb{R} . If $H^1(\mathbb{R}, T) = 0$ then $f_5(G) = 0$. Note that $K = \mathbb{R} \times \mathbb{R}$ or \mathbb{C} . If $K = \mathbb{R} \times \mathbb{R}$, then by Theorem 4.11, $f_5(A) = 0$. Suppose $K = \mathbb{C}$. Note that $L = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $L = \mathbb{R} \times \mathbb{C}$. If $L = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, then by case (i) of proof of Theorem 3.1, $T \cong \mathbb{G}_m \times \mathbb{G}_m$ over \mathbb{R} . Hence \mathbb{R} -rank of $G \geq 2$ and by Lemma 2.2, $f_3(G) = f_5(G) = 0$. Suppose $L = \mathbb{R} \times \mathbb{C}$. Then by case (ii) of proof of Theorem 3.1, $T \cong R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ over \mathbb{R} . Hence \mathbb{R} -rank of $G \geq 1$ and by Lemma 2.2, $f_5(G) = 0$.

Remark 5.10. The real case along with Example 4, leads us to raise the following question: Let L, K be étale algebras of dimension 3, 2 resp. and T be the K -unitary torus associated to the pair (L, K) . Let G be a group of type F_4 defined over k and $T \hookrightarrow G$ over k , then does $H^1(k, T) = 0$ imply $f_5(G) = 0$? Though we have not been able to settle this over an arbitrary field, we can prove a weaker result.

We first state a result which will be required,

Theorem 5.11. (*Knebusch norm principle*) ([15], Chap. VII, Thm. 5.1) *Let K/F be a finite extension of degree n and q be a quadratic form over F . Let $x \in K^*$. If $x \in D_K(q_K)$ then $N_{K/F}(x)$ is a product of n elements of $D_F(q)$. (In particular $N_{K/F}(x) \in [D_F(q)]$). Hence if q is a Pfister form over F and q_K is isotropic, then $N_{K/k}(K^*) \subseteq D_F(q)$.*

Theorem 5.12. *Let L, K be étale algebras over k of dimension 3, 2 resp. and E be the K -unitary algebra and T the K -unitary torus associated with the pair (L, K) . Let G be a group of type F_4 (resp. G_2 or a simple simply connected group of type A_2). Assume there is a k -embedding $T \hookrightarrow G$. If $H^1(\mathbf{U}(E, \tau)) = 0$ then $f_5(A) = 0$ (resp. $\text{Oct}(G)$ splits).*

Proof. Consider the exact sequence

$$1 \longrightarrow \mathbf{U}(E, \tau) \longrightarrow E^* \xrightarrow{N_{E/L}} L^* \longrightarrow 1.$$

The long exact cohomology sequence yields the exact sequence,

$$\mathbf{U}(E, \tau) \longrightarrow E^* \xrightarrow{N_{E/L}} L^* \longrightarrow H^1(\mathbf{U}(E, \tau)) \longrightarrow 1.$$

Hence

$$H^1(\mathbf{U}(E, \tau)) = L^*/N_{E/L}(E^*).$$

Since $H^1(\mathbf{U}(E, \tau)) = 0$, $L^* = N_{E/L}(E^*)$. Let $K = k(\sqrt{\alpha})$ and $q = \langle 1, -\alpha \rangle$. If $K = k \times k$ then by Theorem 4.11, $f_5(G) = 0$. Hence we may assume that K is a field extension. If $L = k \times K_0$, for some quadratic étale algebra K_0 over k , then $H^1(\mathbf{U}(E, \tau)) = k^*/N_{K/k}(K^*) \times K_0^*/N_{K_0 \otimes K/K_0}(K_0 \otimes K)^*$. Since $H^1(\mathbf{U}(E, \tau)) = 0$, $k^* = N_{K/k}(K^*)$. Hence q is universal over k . By Theorem 4.11, q divides $f_5(G)$ and is a subform of $f_5(G)$ (see Theorem 4.6). Hence we have $f_5(G) = 0$. Suppose L is a field extension. Now $q_L = N_{E/L}(E^*)$. Since q_L splits over E , by Theorem 5.11, $N_{E/L}(E^*) = L^* \subseteq D_L(q_L)$. Hence q is universal over an odd degree extension L of k . By Springer's theorem, q is universal over k . Let G be a group of type F_4 . By Theorem 4.11, q divides $f_5(G)$, hence $f_5(G) = 0$. Let G be a group of type G_2 or A_2 . By Theorem 4.3, q divides $f_3(G)$, hence $f_3(G) = 0$. Thus $\text{Oct}(G)$ splits. \square

6 Generation of groups A_2 , G_2 , D_4 and F_4 by rank-2 tori

In this section we deduce the number of rank-2-tori required for the generation of groups of type A_2 , G_2 , and F_4 arising from division algebras and subgroups of type D_4 of $\mathbf{Aut}(A)$ for A an Albert division algebra. Let G be a simple, simply connected group of type A_2 over k . Then the minimum number of maximal tori required to generate G is 2, when k is a perfect field. Let H_i , $i = 1, 2$, be algebraic subgroups of an algebraic group G . By $\langle H_1, H_2 \rangle$ we

will denote the algebraic subgroup of G generated by H_i , $i = 1, 2$. We will often use the Borel-De Siebenthal algorithm. For details see ([3]). Let X and Y be types of root systems. If X is a subsystem of Y , we write $X \subseteq Y$. We begin with,

Lemma 6.1. *Let G be a k -anisotropic, connected, reductive algebraic group over a perfect field k . Let H be a connected subgroup of G . Then H is a reductive, k -anisotropic subgroup.*

Proof. Since G is a k -anisotropic, by Prop. 2.4, $G(k)$ has no non-trivial unipotents. Hence $H(k)$ has no non-trivial unipotents and $R_u(H)(k) = \{1\}$. Since k is perfect, by density of k -points it follows that, $R_u(H) = \{1\}$. \square

Theorem 6.1. *Let k be a perfect infinite field and F be a quadratic étale k -algebra. Let (B, σ) be a degree 3 central division algebra over F with an involution σ of the second kind. Let $G = \mathbf{SU}(B, \sigma)$. Let $E_1, E_2 \subset B$ be F -unitary subalgebras of B such that σ restricts to E_1 and E_2 . Let $\sigma_i = \sigma|_{E_i}$. Assume that $\mathbf{SU}(E_1, \sigma_1) \neq \mathbf{SU}(E_2, \sigma_2)$. Then*

$$G = \langle \mathbf{SU}(E_1, \sigma_1), \mathbf{SU}(E_2, \sigma_2) \rangle.$$

Proof. Let $H = \langle \mathbf{SU}(E_1, \sigma_1), \mathbf{SU}(E_2, \sigma_2) \rangle$. Then H is a connected k -subgroup of G . Since B is a division algebra, G is a k -anisotropic group (see Theorem 4.1). Notice that since $\mathbf{SU}(E_i, \sigma_i)$, $i = 1, 2$ are maximal tori of G , H is a non-toral subgroup. By Lemma 6.1, H is a connected, reductive, k -anisotropic, non-toral subgroup of G . Since G has absolute rank-2, $[H, H]$ is a semisimple group of absolute rank 1 or 2. Hence $[H, H]$ must be of type A_2 , A_1 , $A_1 \times A_1$, G_2 or $B_2 = C_2$. By the Borel-De Siebenthal algorithm, $A_1 \times A_1, B_2 \not\subseteq A_2$. Notice that $G_2 \not\subseteq A_2$ (since Lie algebra of G_2 has dimension 14 whereas the dimension of Lie algebra of A_2 is 8). If $[H, H]$ is of type A_1 , then G has a k -torus S of absolute rank 1, $S \subseteq [H, H]$. Necessarily, $S = \mathbf{M}^{(1)}$, the norm torus of a quadratic extension M/k ([32], Chap.II, §IV, Example 6). But then, S splits over M and hence G becomes isotropic over M . By Prop. 2.8, B remains a division algebra over M . Hence by Theorem 4.1, G remains anisotropic over M , a contradiction. Therefore $[H, H]$ cannot have type A_1 . Hence $[H, H]$ must be of type A_2 . Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Therefore $H = G$. \square

Let G be a group of type G_2 (resp. F_4) defined over k . We now calculate the number of rank-2 tori required to generate G . In ([8], Theorem 3.11, 4.1) we proved that a group of type G_2 is generated by its k -subgroups of type A_1 and a group of type F_4 is generated by its k -subgroups of type A_2 . The results below are continuation of that. We first prove that a group of type G_2 (resp. F_4) is also generated by two k -subgroups of type A_2 (resp. D_4). Using this we deduce that a group of type G_2 (resp. F_4) is generated by three (resp. four) rank-2 tori.

Theorem 6.2. *Let C be an octonion division algebra over k , where k is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by two k -subgroups of type A_2 .*

For the proof of this theorem, we need the following

Proposition 6.1. ([8], Prop. 4.1) *Let C be an octonion division algebra over k . Let $G = \mathbf{Aut}(C)$. Let H be a proper connected reductive, non-toral subgroup of G defined over k . Then $[H, H]$ is of type A_1 , $A_1 \times A_1$ or A_2 .*

We now prove Theorem 6.2.

Proof. Choose quadratic subfields $K_1, K_2 \subset C$ such that $K_1 \cap K_2 = k$. Let $H_i = \mathbf{Aut}(C/K_i)$, $i = 1, 2$. By Theorem 2.3, H_i , $i = 1, 2$, are simple, simply connected subgroups of type A_2 . Let H denote the closed subgroup of G generated by H_i , $i = 1, 2$. By Lemma 6.1, H is a connected, reductive, k -anisotropic, non-toral subgroup of G containing H_i , $i = 1, 2$ properly. By Prop. 6.1, $[H, H]$ is of type $A_1, A_1 \times A_1, A_2$ or G_2 . Now $H_1 = [H_1, H_1] \subseteq [H, H]$. Since H_1 is of type A_2 , by the Borel-De Siebenthal algorithm, $[H, H]$ cannot be of type A_1 or $A_1 \times A_1$. Therefore $[H, H]$ must be of type A_2 or G_2 . If $[H, H]$ is of type A_2 then $[H, H] = H_i$, $i = 1, 2$. Hence $\mathbf{Aut}(C/K_1) = \mathbf{Aut}(C/K_2)$ and hence $\mathbf{Aut}(C/K_1) = \mathbf{Aut}(C/Q)$ where Q denotes the quaternion subalgebra of C generated by K_1 and K_2 . This is a contradiction since $\mathbf{Aut}(C/Q)$ is of type A_1 (Theorem 2.3) while $\mathbf{Aut}(C/K_1)$ is of type A_2 . Hence $[H, H]$ is of type G_2 . Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Therefore $H = G$. \square

From Theorems 6.1, 6.2, we can immediately deduce that when C is an octonion division algebra over a perfect (infinite) field k , then $G = \mathbf{Aut}(C)$ is generated by four k -tori of rank-2. However, we can do better:

Theorem 6.3. *Let C be an octonion division algebra over k , where k is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by three k -tori of rank-2.*

Proof. The algebraic group $G = \mathbf{Aut}(C)$ is a connected, simple algebraic group of type G_2 , in particular, G has absolute rank-2. By Theorem 6.2, G is generated by two subgroups H_i , $i = 1, 2$, of type A_2 with $H_1 \neq H_2$. Choose a maximal k -torus $T \subseteq H_1$ such that $T \not\subseteq H_2$. Let $H = \langle T, H_2 \rangle$ be the (closed) subgroup generated by T and H_2 . Since C is a division algebra, G is k -anisotropic (Prop. 2.6). By Lemma 6.1, H is a connected reductive k -anisotropic non-toral subgroup of G containing H_2 properly. Using same arguments as in Theorem 6.2, it follows that $[H, H]$ must be of type A_2 or G_2 . If $[H, H]$ is of type A_2 , then $[H, H] = H_2$ (since $H_2 = [H_2, H_2] \subseteq [H, H]$ and both are of type A_2). Now $H = [H, H].Z(H)^o = H_2.Z(H)^o$ and $Z(H) = \cap T_i$, T_i 's are maximal tori of H ([2], §13.17, Cor. 2). Since any maximal torus in H_2 is maximal in H we have, $Z(H) \subset H_2$. Hence $H = H_2$ and $T \subseteq H_2$, contradicting the choice of T . Therefore $[H, H]$ is of type G_2 . Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Hence $H = G$. The result now follows since H_2 is itself generated by two rank-2 k -tori. \square

One can derive similar results for groups of type $D_4 \subseteq F_4$ and F_4 . We now find the number of rank-2 tori required to generate $D_4 \subseteq F_4$ and F_4 type groups arising from division algebras.

Theorem 6.4. *Let A be an Albert division algebra over a perfect (infinite) field k and $G = \mathbf{Aut}(A)$. Let $H = \mathbf{Aut}(A/L)$ where L is a 3-dimensional subalgebra of A . Then H is generated by three rank-2 tori over k .*

For the proof of this theorem we need the following,

Lemma 6.2. *Let A be an Albert division algebra over a field k . Let H be a subgroup of G of type D_4 and $H_0 \subseteq H$ be a non-toral reductive k -subgroup. Then $[H_0, H_0]$ is of type A_2 or D_4 .*

Proof. By ([8], Theorem 3.10) $[H_0, H_0]$ is of type $A_2, A_2 \times A_2$ or D_4 . By the Borel-De Siebenthal algorithm, $A_2 \times A_2 \not\subseteq D_4$ and hence $[H_0, H_0]$ must be of type A_2 or D_4 . \square

We now prove Theorem 6.4.

Proof. By ([8] Theorem 3.9), $H = \langle H_1, H_2 \rangle$, where $H_i = \mathbf{Aut}(A/S_i)$ where S_i are 9-dimensional subalgebras of A with $S_1 \cap S_2 = L$. Note that $H_1 \cap H_2 = \{1\}$. By Theorem 2.2, H_i , $i = 1, 2$, is simple, simply connected subgroup of type A_2 . Also H_i , $i = 1, 2$, arise from division algebras.

Claim: We can choose a maximal torus $S \subseteq H_1$ such that $S \not\subseteq \mathbf{Aut}(A, S_2)$.

If not, then $H_1 \subseteq \mathbf{Aut}(A, S_2)$ (since H_1 is generated by its maximal k -tori). Note that $H_2 = \mathbf{Aut}(A/S_2) \subseteq \mathbf{Aut}(A, S_2)$. Hence $H \subseteq \mathbf{Aut}(A, S_2)$, a contradiction, since $D_4 \not\subseteq A_2 \times A_2$. Thus we can choose a maximal k -torus $S \subseteq H_1$ such that $S \not\subseteq \mathbf{Aut}(A, S_2)$. Let $H_0 := \langle S, H_2 \rangle \subseteq H$. Then, by Theorem 6.1, H_0 is generated by three rank 2 k -tori. We will prove that $H_0 = H$. By Lemma 6.1, H_0 is a connected reductive, k -anisotropic, non-toral subgroup of G containing S and H_2 properly. By Lemma 6.2, $[H_0, H_0]$ is of type A_2 or D_4 . If $[H_0, H_0]$ is of type A_2 , then $H_2 = [H_2, H_2] = [H_0, H_0]$ (since H_2 is of type A_2). This shows that H_2 is a normal subgroup of H_0 . Also $S \cap H_2 = \{1\}$, hence $H_0 = \langle S, H_2 \rangle = S.H_2$. Now

$$H_0 = [H_0, H_0].Z(H_0)^o = H_2.Z(H_0)^o.$$

Consider the projection maps τ and τ' given by,

$$Z(H_0)^o \subseteq H_0 = S.H_2 \xrightarrow{\tau} S, H_0 = S.H_2 \xrightarrow{\tau'} H_2.$$

Since $H_0 \neq H_2$, we have $Z(H_0)^o \neq \{1\}$. Since A is a division algebra, $\mathbf{Aut}(A)$ does not have rank-1 k -tori (Theorem 4.1). Hence $Z(H_0)^o$ is a rank-2 k -torus. Since $\tau(Z(H_0)^o)$ is connected, $\tau(Z(H_0)^o) = S$ or $\{1\}$. If $\tau(Z(H_0)^o) = \{1\}$, then $Z(H_0)^o \subseteq H_2$, hence $H_0 = H_2$, a contradiction, since $S \cap H_2 = \{1\}$. Therefore $\tau(Z(H_0)^o) = S$.

Let $H' = \tau'(Z(H_0)^o)$. Note that $1 \in H'$. If $H' = \{1\}$ then $Z(H_0)^o = S$. Since $Z(H_0)^o$ centralizes H_2 , we see that $Z(H_0)^o$ stabilizes A^{H_2} . Therefore $S \subseteq \mathbf{Aut}(A, S_2)$, a contradiction. Hence $H' \neq \{1\}$.

Claim: H' is a rank-2 k -torus of H_2 .

We have, for $s_i h_i \in Z(H_0)^o$, $(s_1 h_1)(s_2 h_2) = s_2(s_1 h_1)h_2 = (s_1 s_2)(h_1 h_2)$. Hence τ' is a homomorphism. It follows that $H' = \tau'(Z(H_0)^o)$ is a k -torus. Now since H_2 does not have any rank-1 k -tori (Theorem 4.1) and $H' \neq \{1\}$, H' is a rank-2 k -torus of H_2 .

Claim: S centralizes H' .

Let $s \in S$ and $h \in H'$. Since $h \in H'$, there exists $s_0 \in S$ such that $s_0 h \in Z(H_0)^o$. Since $s_0 h \in Z(H_0)^o$, we have,

$$shs^{-1} = ss_0^{-1}s_0hs^{-1} = s_0hss_0^{-1}s^{-1} = s_0hs_0^{-1} = h.$$

Hence S centralizes H' and therefore S stabilizes $A^{H'}$. Since $A^{H'} = S_2$, we have $S \subseteq \mathbf{Aut}(A, S_2)$, a contradiction. Hence $[H_0, H_0]$ cannot be of type A_2 . Therefore $[H_0, H_0]$ is of type D_4 . Now $H_0 \subseteq H = [H, H] = [H_0, H_0] \subseteq H_0$. Therefore $H = H_0$ and H is generated by three rank-2 tori over k . \square

Theorem 6.5. *Let A be an Albert division algebra over a perfect (infinite) field k . Then $G = \mathbf{Aut}(A)$ is generated by four rank-2 tori over k .*

We first prove the following lemma,

Lemma 6.3. *Let A be an Albert division algebra over a perfect (infinite) field k and $G = \mathbf{Aut}(A)$. Let $H_i := \mathbf{Aut}(A/L_i) \subseteq G$, $i = 1, 2$, where $L_1 \neq L_2$ are 3-dimensional subalgebras of A . Then G is generated by H_i , $i = 1, 2$.*

Proof. Let $H = \langle H_1, H_2 \rangle$. By Lemma 6.1, H is a connected, reductive, k -anisotropic, non-toral subgroup of G . By ([8], Theorem 3.10), $[H, H]$ is of type A_2 , $A_2 \times A_2$, D_4 or F_4 . Since $D_4 \not\subseteq A_2, A_2 \times A_2$, $[H, H]$ is of type D_4 or F_4 . If $[H, H]$ is of type D_4 , then $H_i = [H_i, H_i] \subseteq [H, H]$, $i = 1, 2$ and H_i , $i = 1, 2$, is of type D_4 , hence $H_i = [H, H]$, $i = 1, 2$, a contradiction since $H_1 \neq H_2$. Therefore $[H, H]$ is of type F_4 . Hence $H = G$. \square

We now give the proof of the above theorem

Proof. By the above lemma, $G = \langle H_1, H_2 \rangle$, $H_i = \mathbf{Aut}(A/L_i)$ where L_i , $i = 1, 2$, are three dimensional subalgebras. Choose a rank-2-torus $T \subseteq H_1$ such that $T \not\subseteq H_2$ (otherwise $H_1 = H_2$ since H_i 's are generated by their rank-2 tori). Let $H = \langle T, H_2 \rangle$. By Lemma 6.1, H is a connected, reductive, k -anisotropic, non-toral subgroup of G . By ([8], Theorem 3.10) the possible types of $[H, H]$ are A_2 , $A_2 \times A_2$, D_4 or F_4 . Now $H_2 = [H_2, H_2] \subseteq [H, H]$. Since H contains H_2 properly, $[H, H]$ cannot be of type A_2 or $A_2 \times A_2$. Suppose $[H, H]$ is of type D_4 . Then $H_2 = [H, H]$. Now $H = [H, H].Z(H)^o = H_2.Z(H)^o$. Since the rank of maximal tori of H and H_2 is four we have, $Z(H)^o = \{1\}$. Hence $H = H_2$, a contradiction. Therefore $[H, H]$ is of type F_4 and $H = G$. \square

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